UNC Charlotte 2004 Comprehensive

March 8, 2004

1. Which of the following lines has a slope that is less than the sum of its x- and y- intercepts?

   (A) $y = 2x + 1$ **(B)** $y = 3x/2 - 1$ **(C)** $y = -4x - 1$

   (D) $y = 4x + 16/3$ **(E)** $y = 3x$

2. Suppose a and b are positive integers for which $(2a+b)^2 - (a+2b)^2 = 9$. What is ab?

   (A) 2 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 24

3. For which of the following values of a does the line $y = a(x-3)$ and the circle $(x-3)^2 + y^2 = 25$ have two points of intersection, one in the 1^{st} quadrant and one in the 4^{th} quadrant?

   (A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** None of A,B,C, and D

4. A two-inch cube ($2 \times 2 \times 2$) of silver weighs 3 pounds and is worth \$320. How much is a three-inch cube of silver worth?

   (A) \$480 **(B)** \$600 **(C)** \$800 **(D)** \$900 **(E)** \$1080

5. The non-zero real numbers a, b, c, d have the property that $\frac{ax+b}{cx+d} = 1$ has no solution in x. What is the value of $\frac{a^2}{a^2+c^2}$?

   (A) 0 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** an irrational number

6. Find an ordered pair (n, m) of positive integers satisfying

$$\frac{1}{n} - \frac{1}{m} + \frac{1}{mn} = \frac{2}{5}.$$

What is mn?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 45

7. A standard deck of playing cards with 26 red and 26 black cards is split into two non-empty piles. In pile A there are four times as many black cards as red cards. In pile B, the number of red cards is an integer multiple of the number of black cards. How many red cards are in Pile B?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

8. The graphs of $x^2 + y^2 = 24x + 10y - 120$ and $x^2 + y^2 = k^2$ intersect when k satisfies $0 \leq a \leq k \leq b$, and for no other positive values of k. Find $b - a$.

(A) 10 **(B)** 14 **(C)** 26 **(D)** 34 **(E)** 144

9. The product of three consecutive non-zero integers is 33 times the sum of the three integers. What is the sum of the digits of this product?

(A) 5 **(B)** 6 **(C)** 12 **(D)** 16 **(E)** 18

10. The three faces of a rectangular box have areas of $40, 45$, and 72 square inches. What is the volume, in cubic inches, of the box?

(A) 300 **(B)** 330 **(C)** 360 **(D)** 400 **(E)** 450

11. It is possible that the difference of two cubes is a perfect square. For example, $28^2 = a^3 - b^3$ for certain positive integers, a and b. In this example, what is $a + b$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

12. Two women and three girls wish to cross a river. Their small rowboat will carry the weight of only one woman or two girls. What is the minimum number of times the boat must cross the river in order to get all five females to the opposite side? At least one person must be in the boat each time it crosses the river.

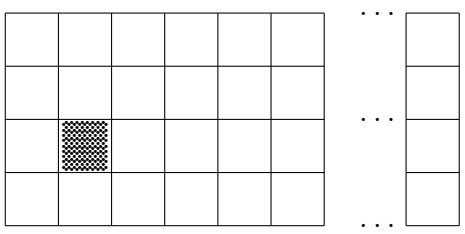
    (A) 9 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 15

13. Maggie has 2 quarters, 3 nickels, and 3 pennies. If she selects 3 coins at random, what is the probability the total value is exactly 35 cents?

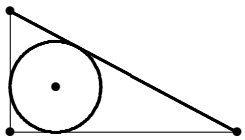
    (A) 3/56 **(B)** 2/28 **(C)** 5/56 **(D)** 3/28 **(E)** 7/56

14. The 4×168 rectangular grid of squares shown below contains a shaded square. Let N denote the number of rectangular subregions that contain the shaded square. What is the sum of the digits of N?

    (A) 6 **(B)** 16 **(C)** 17 **(D)** 26 **(E)** 27



15. Find the radius of the circle inscribed in a triangle whose sides are 8, 15, and 17.



(A) 2.5 **(B)** 2.7 **(C)** 2.9 **(D)** 3.0 **(E)** 3.2

16. A square and an equilateral triangle have the same perimeter. The area of the triangle is 1. What is the area of the square? Express your answer as a decimal to the nearest hundredth.

(A) 1.29 **(B)** 1.30 **(C)** 1.31 **(D)** 1.32 **(E)** 1.33

17. Let $S = 1 + 1/2^2 + 1/3^2 + \cdots + 1/100^2$. Which of the following is true?

(A) $S < 1.40$ **(B)** $1.40 \leq S < 2$ **(C)** $2 \leq S < 4$

(D) $4 \leq S < 100$ **(E)** None of the above

18. The hypotenuse of an isosceles right triangle has a length of b units. What is the area of the triangle.

(A) b **(B)** b^2 **(C)** $b^2/4$ **(D)** $2b^2$ **(E)** $\sqrt{2b^2}$

19. A sphere of radius 2 is centered at $(4, 4, 7)$. What is the distance from the origin $(0, 0, 0)$ to the point on the sphere farthest from the origin?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

20. The six-digit number $5ABB7A$ is a multiple of 33 for digits A and B. Which of the following could be $A + B$?

    (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 14

21. A box of coins contains two with heads on both sides, one standard coin with heads on one side and tails on the other, and one coin with tails on both sides. A coin is randomly selected and flipped twice. What is the probability that the second flip results in heads given that the first flip results in heads?

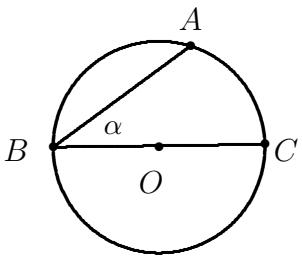
    (A) 0.6 **(B)** 0.7 **(C)** 0.8 **(D)** 0.9 **(E)** 0.95

22. Let V denote the set of vertices of a cube. There are $\binom{8}{3} = 56$ triangles all of whose vertices belong to V. How many of these are right triangles?

    (A) 24 **(B)** 28 **(C)** 32 **(D)** 48 **(E)** 56

23. The circle shown is a unit circle centered at the origin. The segment BC is a diameter and C is the point $(1, 0)$. The angle α has measure 30 degrees. What is the x-coordinate of the point A?

    (A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{\sqrt{3}}{2}$



24. If x, y, and z satisfy

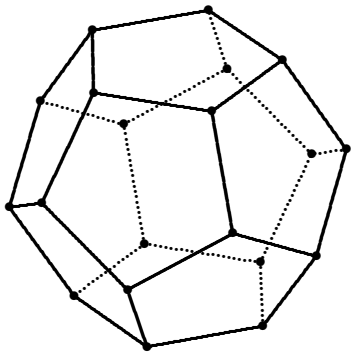
$$\frac{x}{y-6} = \frac{y}{z-8} = \frac{z}{x-10} = 3.$$

    What is the value of $x + y + z =$

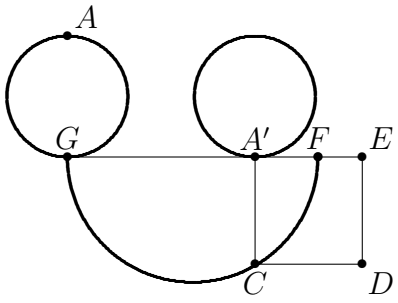
    (A) 24 **(B)** 30 **(C)** 32 **(D)** 36 **(E)** 40

25. A space diagonal of a dodecahedron (ie, a regular 12-sided polyhedron as shown below) is a segment connecting two vertices that do not lie on the same face. How many space diagonals does a dodecahedron have?

(A) 64 **(B)** 90 **(C)** 100 **(D)** 120 **(E)** 150



26. A circle of radius r is 'rolled' horizontally until the point A at the top becomes the bottom A' as shown. The distance $A'F$ is r. The point C is the intersection of the vertical line through A' and the circle with diameter FG. What is the area of the square $A'CDE$?



(A) $3r^2$ **(B)** πr^2 **(C)** $4\pi r^2/3$ **(D)** $2r^2\pi$ **(E)** $4r^2$

27. Let $L(n)$ denote the smallest number of vertical and horizontal line segments needed to construct exactly n non-overlapping unit squares in the plane. Thus, $L(1) = 4, L(2) = 5, L(3) = 6, L(4) = 6$, and $L(100) = 22$. What is $L(2004)$?

    (A) 92 **(B)** 93 **(C)** 94 **(D)** 95 **(E)** 96

28. A convex polyhedron P has 47 faces, 35 of which are triangles, 5 of which are quadrilaterals, and 7 of which are pentagons. How many vertices does P have? Hint: Recall that Euler's theorem provides a relationship among the number f of faces, the number e of edges, and the number v of vertices of a polyhedron:

$$e + 2 = f + v.$$

    (A) 32 **(B)** 34 **(C)** 35 **(D)** 38 **(E)** 40

29. A $4 \times 4 \times 4$ wooden cube is painted on five of its faces and is then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that exactly two of the five visible faces is painted?

    (A) 15/64 **(B)** 17/64 **(C)** 29/128 **(D)** 23/96 **(E)** 71/256

30. Let $D(n)$ denote the leftmost digit of the decimal representation of n. Thus, $D(5^4) = D(625) = 6$. What is $D(6^{2004})$?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 8 **(E)** 9