# UNC Charlotte 2004 Comprehensive with solutions

March 8, 2004

1. Which of the following lines has a slope that is less than the sum of its $x$- and $y$- intercepts?

   **(A)** $y = 2x + 1$ **(B)** $y = 3x/2 - 1$ **(C)** $y = -4x - 1$

   **(D)** $y = 4x + 16/3$ **(E)** $y = 3x$

   **Solution:** C. Check these to see that only $y = -4x - 1$ satisfies the condition.

2. Suppose $a$ and $b$ are positive integers for which $(2a + b)^2 - (a + 2b)^2 = 9$. What is $ab$?

   **(A)** 2 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 24

   **Solution:** A. Expand and cancel to get $4a^2 + 4ab + b^2 - a^2 - 4ab - 4b^2 = 9$, so $3a^2 - 3b^2 = 9$. This can happen only if $(a - b)(a + b) = 3$, which is true in case $a - b = 1$ and $a + b = 3$ since $a + b$ cannot be negative. It follows that $a = 2$ and $b = 1$, so $ab = 2$.

3. For which of the following values of $a$ does the line $y = a(x - 3)$ and the circle $(x - 3)^2 + y^2 = 25$ have two points of intersection, one in the $1^{\text{st}}$ quadrant and one in the $4^{\text{th}}$ quadrant?

   **(A)** $-1$ **(B)** 0 **(C)** 1 **(D)** 2 **(E)** None of A,B,C, and D

   **Solution:** D. The line goes through the center of the circle, $(3, 0)$ so it must intersect the circle twice. For $a = -1$ the intersection includes a point in the second quadrant. For $a = 2$, the line is $y = 2x - 6$, so $(x - 3)^2 + (2x - 6)^2 = 25$, which becomes $(x - 3)^2 = 5$, which has two positive solutions. For $a = 0$ the intersection includes only points on the $x$-axis. For $a = 1$, the intersection includes a point of the third quadrant.

   Alternate Solution: The intersection of the circle with the $y$-axis is a pair of points $(0, 4)$ and $(0, -4)$. The slope of the line through the center and $(0, 4)$ is $-4/3$ and the slope of the line thought the center and the point $(0, -4)$ is $4/3$. The lines with slopes 1, 0 and $-1$ cross the $y$-axis between $(0, 4)$ and $(0, -4)$, so cannot intersect the circle in both the first and fourth quadrants. On the other hand, a line through the center whose slope has absolute values greater than $4/3$ intersects the circle in both of these quadrants. Thus $a = 2$ works.

4. A two-inch cube $(2 \times 2 \times 2)$ of silver weighs 3 pounds and is worth \$320. How much is a three-inch cube of silver worth?

   **(A)** \$480 **(B)** \$600 **(C)** \$800 **(D)** \$900 **(E)** \$1080

   **Solution:** E. The value of a $3 \times 3 \times 3$ cube is $(27/8)(320) = 27 \cdot 40 = 1080$ dollars.

5. The non-zero real numbers $a, b, c, d$ have the property that $\frac{ax+b}{cx+d} = 1$ has no solution in $x$. What is the value of $\frac{a^2}{a^2+c^2}$?

   **(A)** 0 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** an irrational number

   **Solution:** B. Contestant Anders Kaseorg has pointed out to us that answer E. is also correct. Note that if $a \neq c, b = ka$, and $d = kc$, then $\frac{ax+b}{cx+d} = \frac{ax+ka}{cx+kc} = \frac{a+k}{c+k} \neq 1$. Let $a = 1$ and $c = \pi$. Then $\frac{a^2}{a^2+c^2}$ is irrational. The posers intended to eliminate this counterexample by imposing the usual non-zero determinant condition $ad - bc \neq 0$. With this extra condition, the equation is solvable if $ax + b = cx + d$ which is equivalent to $(a - c)x = d - b$. This has a solution unless $a = c$. Therefore $a = c$ and it follows that $\frac{a^2}{a^2+c^2} = \frac{1}{2}$.

6. Find an ordered pair $(n, m)$ of positive integers satisfying

$$\frac{1}{n} - \frac{1}{m} + \frac{1}{mn} = \frac{2}{5}.$$

What is $mn$?

**(A)** 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 45

**Solution:** B. By multiplying each fraction by $5mn$, we can transform the equation into the equivalent $2mn = 5(m - n + 1)$. Notice that this implies that either $m$ or $n$ is a multiple of 5. This leads us to try various multiples of 5 for $m$. Alternatively, transform the equation further to get

$$2mn - 5m + 5n - 5 = 0.$$

Subtract $-15/2$ from both sides so that we can factor to get

$$m(2n - 5) + 5(2n - 5)/2 = -15/2.$$

Massage this to get $(2n-5)(2m+5) = -15$. Since $m$ is positive, $2m+5 > 0$. Therefore, $2n - 5 < 0$. The only values of $n$ making $2n - 5 < 0$ are $n = 1$ and $n = 2$. The choice $n = 1$ requires $m = 0$. So the solution $m = 5, n = 2$ is unique. Alternatively, clear the fractions to get $5(m+1) - 5n = 5m - 5n + 5 = 2mn$. Since $m$ and $n$ are positive integers, the same is true for $2mn$. Thus $m + 1 > n$ and $5 - 5n \leq 0$. If $n = 1$, we would have $5m = 2m$ which is impossible. Thus $n > 1$ and we have $5m > 2mn$ so $5 > 2n$. Therefore the only possible value for $n$ is 2. Now simply solve for $m$: $5m - 10 + 5 = 4m$, so $m = 5$ and $mn = 10$.

7. A standard deck of playing cards with 26 red and 26 black cards is split into two non-empty piles. In pile A there are four times as many black cards as red cards. In pile B, the number of red cards is an integer multiple of the number of black cards. How many red cards are in Pile B?

   **(A)** 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

   **Solution:** C. Let $r_1$ and $b_1$ denote the number of red and black cards in pile A, so the numbers in pile B must be $r_2 = 26 - r_1$ and $b_2 = 26 - b_1$. Also, the numbers $r_1$ and $b_1$ satisfy $b_1 = 4r_1$ and $26 - r_1 = k(26 - b_1)$ for some integer $k$. Combining these equations and solving for $r_1$, we find that

$$r_1 = \frac{26(k-1)}{4k-1}.$$

   Trying values of $k$ starting at 2, we finally find that when $k = 10$, $r_1$ is an integer. Solving this we get $r_1 = \frac{9 \cdot 26}{39} = 6$ and thus $r_2 = 20$. Alternate Solution: Let $r$ be the number of red cards in stack A and let $b$ be the number of black cards in stack A. Then $4r = b > 0$ and $26 - r = p(26 - b)$ for some integer $p$. Since $b$ is even and $p$ is an integer, $r$ must be even. Thus $b$ is a positive integer multiple of 8. The only such numbers less than or equal to 26 are 8, 16 and 24. Having $b = 8$ requires $r = 2$. For this pair $24 = 26 - 2$ is not an integer multiple of $18 = 26 - 8$. Having $b = 16$ requires $r = 4$. For this pair $22 = 26 - 4$ is not an integer multiple of $10 = 26 - 16$. But for $b = 24$ we have $r = 6$ and $20 = 26 - 6 = 10(2) = 10(26 - 24)$.

   For yet another approach, substitute $4r$ for $b$ into the second equation and solve for $r$ to get $r = 26(p-1)/(4p-1)$. If the denominator is not an integer multiple of 13, then $r$ will be. But that is impossible since $4r = b \leq 26$. Thus $4p - 1$ must be an odd integer multiple of 13. The smallest positive integer multiple of 13 that is one less than a multiple of 4 is 39. This occurs when $p = 10$. The corresponding value for $r$ is 6. This yields $b = 24$. [Here is a formal proof that this is the only positive integer solution: If $0 < r < s$ are integer multiples of 13 that are both one less than a multiple of 4, then $s - r$ is both an integer multiple of 13 and an integer multiple of 4. Thus every positive integer multiple of 13 that is both one less than a multiple of 4 and strictly larger than 39 is of the form $39 + 4 \cdot 13k = 39 + 52k$ for some positive integer $k$. Let $q = 10 + 13k$. Then $4q - 1 = 39 + 52k$. Substitute $q$ in for $p$ in the fraction $26(p-1)/(4p-1)$ and then replace $q$ by $10 + 13k$. The resulting fraction is $26(9+13k)/(39+52k)$ which reduces to $(18 + 26k)/(3 + 4k) = 6 + (2k/(3 + 4k)) > 6$ since $k > 0$.

Obviously, $2k/(3+4k)$ is never an integer when $k$ is a positive. So the only solution is when $p = 10$.]

8. The graphs of $x^2 + y^2 = 24x + 10y - 120$ and $x^2 + y^2 = k^2$ intersect when $k$ satisfies $0 \le a \le k \le b$, and for no other positive values of $k$. Find $b - a$.

   **(A)** 10 **(B)** 14 **(C)** 26 **(D)** 34 **(E)** 144

   **Solution:** B. The first circle $(x-12)^2 + (y-5)^2 = 49$ is centered at $(12, 5)$ and has radius 7, while the second is centered at $(0, 0)$ and has radius $k$. The two circle intersect when $6 \le k \le 20$, so $b - a = 14$.

   Alternatively, let $C$ be the circle given by the equation $x^2+y^2 = 24x+10y-120$. An equivalent equation for $C$ is $(x-12)^2 + (y-5)^2 = 49$. So the radius of $C$ is 7 and the center is $(12, 7)$. Let $L$ be the line through the origin and the point $(12, 5)$, the center of $C$. Since the radius of $C$ is less than 12, the line $L$ intersects $C$ at two points in the first quadrant, call them $P$ and $Q$ and assume the $x$-coordinate of $P$ is smaller than the $x$-coordinate of $Q$. The point $P$ must be on the circle with center at $(0, 0)$ and radius $a$ and the point $Q$ must be on the circle with center at $(0, 0)$ and radius $b$. Since $P$ and $Q$ are on the circle $C$, on a line through the center of $C$ and on the same line through the center of the other two circles, the distance between $P$ and $Q$ is $14 = b - a$.

9. The product of three consecutive non-zero integers is 33 times the sum of the three integers. What is the sum of the digits of this product?

   **(A)** 5 **(B)** 6 **(C)** 12 **(D)** 16 **(E)** 18

   **Solution:** E. If the three integers are denoted $n-1, n$, and $n+1$, then their sum is $3n$ and their product is $n(n^2-1)$. Thus $n(n^2-1) = 33 \cdot 3n = 99n$, from which it follows that $n^2 - 1 = 99$ and so $n = 10$. The product is 990 and the sum of the digits is $9 + 9 + 0 = 18$.

10. The three faces of a rectangular box have areas of $40, 45$, and $72$ square inches. What is the volume, in cubic inches, of the box?

    **(A)** 300 **(B)** 330 **(C)** 360 **(D)** 400 **(E)** 450

    **Solution:** C. Let $x, y$, and $z$ denote the dimensions of the box. Then, $x^2y^2z^2 = xy \cdot xz \cdot yz = 40 \cdot 45 \cdot 72 = 129,600$. Therefore, $xyz = \sqrt{129,600} = 360$. Alternatively, let $x$, $y$ and $z$ be the dimensions with $xy = 40$, $xz = 45$ and $yz = 72$. Solve directly for $x$ by solving $yz = 72$ for $z$ and then substituting into $xz = 45$ to get $72x/y = 45$. Thus $y = 72x/45$. Substitute this into $xy = 40$ to obtain $72x^2/45 = 40$. This yields $x^2 = 25$, so $x = 5$. Thus the volume is $x(yz) = 5(72) = 360$. [Or solve for $y$ or $z$.]

11. It is possible that the difference of two cubes is a perfect square. For example, $28^2 = a^3 - b^3$ for certain positive integers, $a$ and $b$. In this example, what is $a + b$?

    **(A)** 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

    **Solution:** C. Since $28^2 = a^3 - b^3$ is an even number, $a$ and $b$ must have the same parity (either both are odd or both are even). In either case, $a-b$ is even. Next note that $b \neq 1$ since $28^2+1^3$ is not a perfect cube. Therefore $b \geq 2$. Note that $28^2 = a^3-b^3 = (a-b)(a^2+ab+b^2)$. Now $a-b < 8$ because $a-b \geq 8$ implies that $a^2 + ab + b^2 \leq 28^2/8 = 98$, but $a \geq b+8 \geq 10$ implies $a^2 + ab + b^2 \geq 100$. Thus $a - b$ is an even factor of $28^2$ that is less than 8. The only possibilities are 2 and 4. Trying $a - b = 2$ yields the quadratic $a^2 + ab + b^2 = 196$ which reduces to $(b+2)^2 + (b+2)b + b^2 = 196$, which has no integer solutions. Trying $a - b = 4$ yields $a = b + 4$ and so $(b + 4)^2 + (4 + b)b + b^2 = 28^2 \div 4 = 196$. This reduces to $3b^2 + 12b - 180 = 0$, which can be factored to yield roots of $b = -10$ and $b = 6$. It follows that $a = 10$ and $a + b = 16$.

    Alternatively, one could build a table with values $784 + n^3$ for integer values of $n$ and see when a perfect cube comes up. Yet another alternative is to rewrite the equation as $28^2 + b^3 = a^3$. Since $a$ and $b$ are positive integers, $a^3 > 28^2 = 784$. Note that $1000 = 10^3$ is larger than 784 and $729 = 9^3$ is smaller. Thus the smallest integer that might possibly work for $a$ is $a = 10$. This actually works since $1000 - 784 = 216 = 6^3$. So if there is a correct single answer, it must be that $a + b = 16$.

12. Two women and three girls wish to cross a river. Their small rowboat will carry the weight of only one woman or two girls. What is the minimum number of times the boat must cross the river in order to get all five females to the opposite side? At least one person must be in the boat each time it crosses the river.

    **(A)** 9 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 15

    **Solution:** C. It takes 11 trips back and forth. We can write this as follows: gg g w g gg g w g gg g gg where the first, third, fifth, etc symbol represents a trip from the starting bank to the destination bank, and the symbols in the even positions represent the return trips.

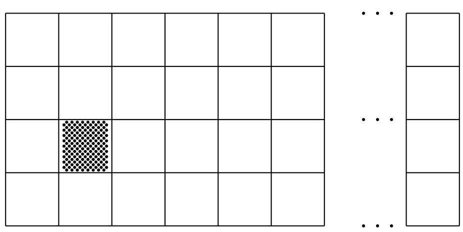
13. Maggie has 2 quarters, 3 nickels, and 3 pennies. If she selects 3 coins at random, what is the probability the total value is exactly 35 cents?

    **(A)** 3/56 **(B)** 2/28 **(C)** 5/56 **(D)** 3/28 **(E)** 7/56

    **Solution:** D. Since the total value of the three coins is exactly 0.35, the coins must be a quarter and two nickels. There are 6 ways to pick a quarter and two nickels out of a total of $\binom{8}{3} = 56$ ways to select three coins, so the probability is $6/56 = 3/28$.

14. The $4 \times 168$ rectangular grid of squares shown below contains a shaded square. Let $N$ denote the number of rectangular subregions that contain the shaded square. What is the sum of the digits of $N$?

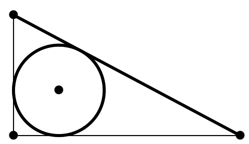
    **(A)** 6 **(B)** 16 **(C)** 17 **(D)** 26 **(E)** 27



**Solution:** A. Each rectangular region is determined by two pairs of horizontal and vertical lines. In order that the shaded square is inside the region, the lower bounding line must be one of the bottom two, the upper boundary line must be one of the top three, the left side must be one of the two left-most vertical lines and the right boundary line must be one of the 167 vertical lines to the right on the shaded square. Thus the number of regions is $N = 2 \cdot 3 \cdot 2 \cdot 167 = 2004$ and the sum of the digits is 6.

Alternatively, count based on the height of the rectangle and position of bottom left square from the grid that is part of the rectangle. A rectangle of length one must have its bottom left square in the second column, and one of length 168 must have its bottom left square in the first column. All other lengths may have bottom left square in either of the first two columns. So for rectangles of height 1, there is one of length 1, one of length 168, and two each of all 166 other lengths. The same count works for rectangles of height 4 – one of length 1, one of length 168 and two each of all 166 other lengths. For rectangles of height 2, there are two of length 1, two of length 168, and four each of all 166 other lengths. The same count works for rectangles of height 3 – two of length 1, two of length 168, and four each of all 166 other lengths. Thus there are six rectangles of length 1, six of length 168, and twelve of each other length. Therefore the total number of rectangles is $2004 = 6 + 6 + 12 \cdot 166$. The sum of the digits is 6.

15. Find the radius of the circle inscribed in a triangle whose sides are 8, 15, and 17.



**(A)** 2.5 **(B)** 2.7 **(C)** 2.9 **(D)** 3.0 **(E)** 3.2

**Solution:** D. Let $r$ denote the radius. Since the triangle is a right triangle, its area $A$ is easily computed in two ways. First $A = (1/2)bh = (1/2)15 \cdot 8 = 60$. Also, we can break the triangle into three triangles by connecting the center of the circle with each of the vertices. Then we get $A$ by adding the areas of these triangles together:

$$\frac{15r}{2} + \frac{8r}{2} + \frac{17r}{2} = \frac{40r}{2}.$$

It follows that $40r = 2 \cdot 60$ and $r = 3$.

Alternatively, the distance from a point $(s, t)$ to a line given by the equation $Ax + By + C = 0$ is $|As + Bt + C|/\sqrt{A^2 + B^2}$. Position the triangle in the plane so that the vertices are $(0, 0)$, $(0, 8)$ and $(15, 0)$. Then the coordinates of the center of the circle are $(r, r)$ where $r$ is the radius. An equation for the line through $(0, 8)$ and $(15, 0)$ is $8x + 15y - 120 = 0$. By the formula, the distance from $(r, r)$ to this line is $|8r + 15r - 120|/\sqrt{8^2 + 15^2} = |120 - 23r|/17$. Of course this distance is also equal to $r$. In this case, $120 - 23r$ is positive. So $120 - 23r = 17r$, and $r = 3$. (The solution of $23r - 120 = 17r$ is $r = 20$ which is too big to be the radius.)

16. A square and an equilateral triangle have the same perimeter. The area of the triangle is 1. What is the area of the square? Express your answer as a decimal to the nearest hundredth.

**(A)** 1.29 **(B)** 1.30 **(C)** 1.31 **(D)** 1.32 **(E)** 1.33

**Solution:** B. Suppose the triangle has perimeter $12a$. Then each side of the square has side $3a$. The area of the triangle is given by $1 = \frac{(4a)^2\sqrt{3}}{4}$ from which it follows that $a^2 = 1/(4\sqrt{3})$. Therefore the area of the square is $9a^2 = \frac{9}{4\sqrt{3}} = \frac{3\sqrt{3}}{4} \approx 1.299 \approx 1.30$.

17. Let $S = 1 + 1/2^2 + 1/3^2 + \cdots + 1/100^2$. Which of the following is true?

    **(A)** $S < 1.40$ **(B)** $1.40 \le S < 2$ **(C)** $2 \le S < 4$

    **(D)** $4 \le S < 100$ **(E)** None of the above

    **Solution:** B. Since $1 + 1/2^2 + 1/3^2 + 1/4^2 > 1.4$, we can rule out option A. On the other hand, $S =$

    $$\sum_{n=1}^{100} \frac{1}{n^2} \le \sum_{n=1}^{100} \frac{1}{n^2 - (.5)^2} = \sum_{n=1}^{100} \left( \frac{1}{n - .5} - \frac{1}{n + .5} \right) = \frac{1}{1 - \frac{1}{2}} - \frac{1}{100 + \frac{1}{2}} < \frac{1}{1 - \frac{1}{2}} = 2.$$

    Alternatively, the sum $1 + 1/4 + 1/9 + 1/16$ is between 1.42 and 1.43. So the sum is larger than 1.4. For the rest use some rather crude estimates based on the noticing that $1/n^2 + 1/(n+1)^2 + \cdots + 1/(2n-1)^2$ is less than $n \cdot (1/n^2) = 1/n$ and greater than $n \cdot (1/(2n)^2) = 1/4n$. So $0.05 = 1/20 < 1/5^2 + 1/6^2 + \cdots 1/9^2 < 1/5 = 0.2$, $1/40 = 0.25 < 1/10^2 + 1/11^2 + \cdots + 1/19^2 < 1/10 = 0.1$, $1/80 = 0.0125 < 1/20^2 + \cdots + 1/39^2 < 1/20 = 0.05$, $1/160 = 0.00625 < 1/40^2 + \cdots + 1/79^2 < 1/40 = 0.025$ and $21 \cdot (1/100^2) = 0.0021 < 1/80^2 + \cdots + 1/100^2 < 21 \cdot (1/80^2) = 0.00328125$. Thus the entire sum is between $1.42 + 0.05 + 0.025 + 0.0125 + 0.00625 + 0.0021 = 1.50335$ and $1.43 + 0.2 + 0.1 + 0.05 + 0.025 + 0.00328125 = 1.80828125 < 2$.

18. The hypotenuse of an isosceles right triangle has a length of $b$ units. What is the area of the triangle.

    **(A)** $b$ **(B)** $b^2$ **(C)** $b^2/4$ **(D)** $2b^2$ **(E)** $\sqrt{2b^2}$

    **Solution:** C. Since the hypotenuse is $b$, the Pythagorean Theorem gives sides of length $b/\sqrt{2}$. Since we have a right triangle, the area is half the product of the lengths of the two legs, or $\frac{1}{2} \cdot b/\sqrt{2} \cdot b/\sqrt{2} = b^2/4$.

19. A sphere of radius 2 is centered at $(4, 4, 7)$. What is the distance from the origin $(0, 0, 0)$ to the point on the sphere farthest from the origin?

    **(A)** 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

    **Solution:** D. The distance is 2 more than the distance to the center of the sphere, which is $2 + \sqrt{4^2 + 4^2 + 7^2} = 2 + \sqrt{81} = 11$.

20. The six-digit number $5ABB7A$ is a multiple of 33 for digits $A$ and $B$. Which of the following could be $A+B$?

    **(A)** 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 14

    **Solution:** B. A rule for divisibility by 11 is that the alternating sum of the digits should be a multiple of 11. Thus, divisibility by 11 implies that $2A + B - (B + 12) = 2A - 12$ is zero or $\pm 11$. Since $2A - 12$ is even, it follows that $2A - 12 = 0$ and that $A = 6$. So the number $56BB76$ is a multiple of 3, and this implies that $B = 0, 3, 6$ or 9. So $A + B$ could be $6, 9, 12$, or 15 only. Alternatively, the sum of the digits is $12+2A+2B$. If the number is divisible by 33, then it must be divisible by 3. Thus 3 divides $2A+2B$. Since 2 and 3 are primes, 3 must divide $A+B$. For the choices given, the factor of 11 is not important as only one of the choices is a multiple of 3.

21. A box of coins contains two with heads on both sides, one standard coin with heads on one side and tails on the other, and one coin with tails on both sides. A coin is randomly selected and flipped twice. What is the probability that the second flip results in heads given that the first flip results in heads?

    **(A)** 0.6 **(B)** 0.7 **(C)** 0.8 **(D)** 0.9 **(E)** 0.95

    **Solution:** D. The first flip of heads comes from a two-headed coin with probability 0.8 and the flip comes from a normal coin with probability 0.2, so the probability that the second flip results in heads is $0.8 \cdot 1 + 0.2 \cdot .5 = 0.9$. Alternatively, the conditional probability of event $A$ occurring given that event $B$ equals the probability of both $A$ and $B$ occurring divided by the probability event $B$ occurs. So first calculate the probability both tosses are heads, then divide by the probability the first toss is heads. The probability a particular coin is chosen is $1/4$. Thus the probability both tosses are heads is $(1/4) \cdot 1 + (1/4) \cdot 1 + (1/4) \cdot (1/4) + (1/4) \cdot 0 = 9/16$. For the first toss, the probability of heads is $(1/4) \cdot 1 + (1/4) \cdot 1 + (1/4) \cdot (1/2) = 5/8$. Thus the conditional probability is $(9/16)/(5/8) = 9/10$.

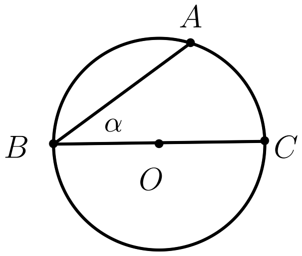
22. Let $V$ denote the set of vertices of a cube. There are $\binom{8}{3} = 56$ triangles all of whose vertices belong to $V$. How many of these are right triangles?

    **(A)** 24 **(B)** 28 **(C)** 32 **(D)** 48 **(E)** 56

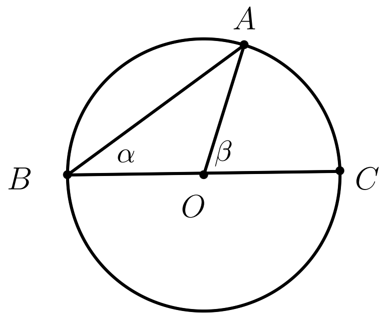
    **Solution:** D. There are 24 right triangles with edge lengths $1, 1, \sqrt{2}$; 24 right triangles with edge lengths $1, \sqrt{2}, \sqrt{3}$; and 8 equilateral triangles with edge lengths $\sqrt{2}, \sqrt{2}, \sqrt{2}$.

23. The circle shown is a unit circle centered at the origin. The segment $BC$ is a diameter and $C$ is the point $(1, 0)$. The angle $\alpha$ has measure 30 degrees. What is the $x$-coordinate of the point $A$?

**(A)** $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{\sqrt{3}}{2}$




**Solution:** C. Recall a fact from geometry: the angle $\beta$ (see below) is twice the measure of the given angle. Then $x = \cos 60 = 1/2$. Otherwise, suppose that the coordinates of A are $(x, y)$. Then $y = (1 + x)\tan\alpha = \frac{1+x}{\sqrt{3}}$ and $x^2 + y^2 = 1$. Substituting $y$ from the first equation into the second, we have $x^2 + \left(\frac{1+x}{\sqrt{3}}\right)^2 = 1$. After squaring and simplifying, we find $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$. The only positive root is $\frac{1}{2}$. Alternatively, note that angle $ACO$ is $60°$. It follows that $\triangle ACO$ is equilateral. Therefore, the projection of point $A$ onto the $x$-axis is the midpoint of $OC$, namely, 1/2. Therefore the $x$-coordinate of $A$ os 1/2.




Alternate Solution: Let $(r, s)$ be the coordinates of $A$. Then $r^2 + s^2 = 1$. Since $O$ is the center of the circle , the segments $\overline{OB}$, $\overline{OA}$ and $\overline{OC}$ all have length 1. Also $\angle ABC$ is a right angle and the measure of $\angle OCA$ is $60°$. Thus $\triangle\, AOC$

is equilateral. Then $r^2 + s^2 = 1$ and $(r-1)^2 + s^2 = 1$. Squaring out $(r-1)^2$ and subtracting the second equation from the first yields $2r - 1 = 0$. Thus $r = 1/2$.

24. If $x, y$, and $z$ satisfy

$$\frac{x}{y-6} = \frac{y}{z-8} = \frac{z}{x-10} = 3.$$

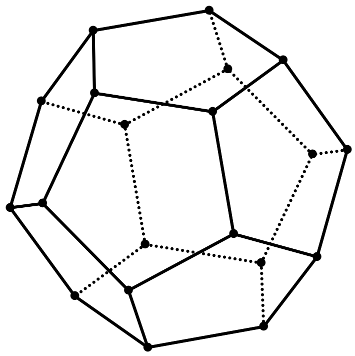
What is the value of $x + y + z =$

**(A)** 24 **(B)** 30 **(C)** 32 **(D)** 36 **(E)** 40

**Solution:** D. Add together the three equations, $x = 3(y-6)$, $y = 3(z-8)$, and $z = 3(x-10)$ to get $x + y + z = 3(x + y + z - 24)$, from which it follows that $x + y + z = 36$. This does not prove that such $x, y, z$ exist. However, you can check to see that $x = 180/13$, $y = 138/13$ and $z = 150/13$ does work.

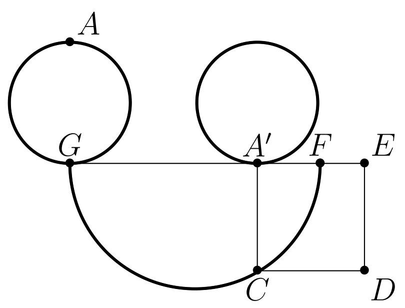
25. A space diagonal of a dodecahedron (ie, a regular 12-sided polyhedron as shown below) is a segment connecting two vertices that do not lie on the same face. How many space diagonals does a dodecahedron have?

**(A)** 64 **(B)** 90 **(C)** 100 **(D)** 120 **(E)** 150



**Solution:** C. Each of the 20 vertices is adjacent to three others, belongs to the same pentagonal face and is not adjacent to six others, and is not on a space diagonal with itself. So each vertex is the endpoint of 10 space diagonals. Since there are 20 vertices, there are 200 such space diagonal endpoints, and therefore 100 space diagonals. Alternatively, first note that there are 30 edges and there are $190 = \binom{20}{2}$ distinct line segments formed by connecting pairs of vertices. Thus the total number of diagonals is $160 = 190 - 30$. Each face has 5 edges and there are $10 = \binom{5}{1}$ distinct line segments connecting pairs of vertices in any one face. Thus each face has $5 = 10 - 5$ diagonals. These are the non-space diagonals. Since there are twelve faces, the number of non-space diagonals is $60 = 12 \cdot 5$. Therefore the number of space diagonals is $100 = 160 - 60$.

26. A circle of radius $r$ is 'rolled' horizontally until the point $A$ at the top becomes the bottom $A'$ as shown. The distance $A'F$ is $r$. The point $C$ is the intersection of the vertical line through $A'$ and the circle with diameter $FG$. What is the area of the square $A'CDE$?




**(A)** $3r^2$ **(B)** $\pi r^2$ **(C)** $4\pi r^2/3$ **(D)** $2r^2\pi$ **(E)** $4r^2$

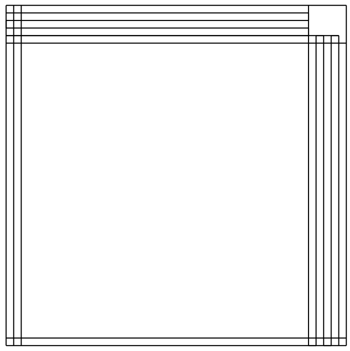
**Solution:** B. Let $x$ denote $A'C$. Because triangle $A'CF$ is similar to triangle $A'GC$, $x$ is the geometric mean of $A'G$ and $A'F$; that is, $x^2 = \pi r \cdot r = \pi r^2$. Therefore the area of the square is $\pi r^2$.

Alternatively, first consider a more general problem where the distance between $A'$ and $F$ is $s$ with $s$ not necessarily equal to $r$. Then the distance between $G$ and $A'$ is $\pi s$ since the circle has made half a revolution. Let $x$ be the distance from $A'$ to $C$, $y$ the distance from $G$ to $C$ and $z$ the distance from $F$ to $C$. Since $C$, $G$ and $F$ are on a circle with $G$ and $F$ and endpoints of a diameter, $\angle GCF$ is a right angle. Since $\angle GA'C$ is also a right angle, $x^2 + \pi^2 s^2 = y^2$, $x^2 + r^2 = z^2$ and $y^2 + z^2 = (\pi s + r)^2 = \pi^2 s^2 + 2\pi sr + r^2$. Adding the first two together yields $2x^2 + \pi^2 s^2 + r^2 = \pi^2 s^2 + 2\pi sr + r^2$. Thus the area of square in the general setting is $\pi sr$. Since $s = r$ in this problem, the area of the square is $\pi r^2$.

27. Let $L(n)$ denote the smallest number of vertical and horizontal line segments needed to construct exactly $n$ non-overlapping unit squares in the plane. Thus, $L(1) = 4, L(2) = 5, L(3) = 6, L(4) = 6$, and $L(100) = 22$. What is $L(2004)$?

    **(A)** 92 **(B)** 93 **(C)** 94 **(D)** 95 **(E)** 96

**Solution:** A. We can construct $45^2 = 2025$ unit squares with $45+1+45+1 = 92$ segments, half horizontal and half vertical. Note that $2004 = 45^2 - 25 + 4$, so there is hope to accomplish the construction with 92 lines. Let 84 of the lines have length 45; 4 have length 41; 3 have length 40, and 1 with length 44. On the other hand, the maximum number of unit squares that can be constructed using 91 segments is $44 \cdot 45 = 1980$, so 91 lines is not sufficient.



Alternate Solution: An "additive" approach. For a perfect square $m^2$, the minimum number of lines needed is $2 \cdot (m + 1)$. The minimum is obtained by making an $m \times m$ grid using $m+1$ vertical line segments and $m+1$ horizontal line segments. To make $m^2 + 1$ through $m^2 + m$ unit squares, simply add one more vertical segment and extend the appropriate number of consecutive horizontal ones to get the required number. For more than $m^2 + m$ and up to $(m + 1)^2$, add one horizontal segment to the $m + 1 \times m$ grid and extend the appropriate number of consecutive vertical segments. Since $\sqrt{2004} \approx 44.77$, the largest perfect square that is less than or equal to 2004 is 44. The difference $2004 - 44^2 = 68$ is greater than 44, so the minimum number of lines for 2004 is $92 = 2 \cdot (45 + 1)$, the same minimum for $45^2$.

28. A convex polyhedron $P$ has 47 faces, 35 of which are triangles, 5 of which are quadrilaterals, and 7 of which are pentagons. How many vertices does $P$ have? Hint: Recall that Euler's theorem provides a relationship among the number $f$ of faces, the number $e$ of edges, and the number $v$ of vertices of a polyhedron:

$$e + 2 = f + v.$$

**(A)** 32 **(B)** 34 **(C)** 35 **(D)** 38 **(E)** 40

**Solution:** C. The number of edges is $\frac{1}{2}(35 \cdot 3 + 5 \cdot 4 + 7 \cdot 5) = 80$. By Euler's formula, $e + 2 = f + v = 82$, it follows that there are $82 - 47 = 35$ vertices. This polyhedron is called a *gyroelongated pentagonal cupolarotunda* (J47).

29. A $4 \times 4 \times 4$ wooden cube is painted on five of its faces and is then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that exactly two of the five visible faces is painted?

**(A)** 15/64 **(B)** 17/64 **(C)** 29/128 **(D)** 23/96 **(E)** 71/256

**Solution:** D. There are four cubes with paint on three adjacent faces, 20 cubes with paint on two (adjacent) faces, and 28 cubes with only one painted face. The other 12 cubes have no painted faces. A cube with three painted faces has probability 1/2 of landing so that two of the faces can be seen, and a cube with 2 painted faces has probability 2/3 of landing so that two painted faces are showing. Thus the probability we seek is $(4/64)\cdot(1/2)+(20/64)\cdot(2/3) = 23/96$.

30. Let $D(n)$ denote the leftmost digit of the decimal representation of $n$. Thus, $D(5^4) = D(625) = 6$. What is $D(6^{2004})$?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 8 **(E)** 9

**Solution:** B. Note that $\log 6^{2004} = 2004 \log 6 = 1559.415\ldots$ which means that $6^{2004}$ is a 1560 digit number that begins with the same digits as $10^{.415\cdots} = 2.6007\ldots$, so we have $6^{2004} = 26007\ldots$ and $D(6^{2004}) = 2$.