UNC Charlotte 2006 Algebra

March 6, 2006

1. The name 'algebra' derives its origin from

   (A) a middle eastern terrorist organization **(B)** an insidious green plant

   (C) a piece of the female wardrobe **(D)** evidence of bad dental hygiene

   (E) an arabic text on 'Restoring and Simplification'

   Solution: E.

2. What is the length of the interval of solutions to the inequality $1 \leq 3-4x \leq 11$?

   (A) 1.75 **(B)** 2.00 **(C)** 2.25 **(D)** 2.50 **(E)** 3.25

   Solution: D. Subtract 3 from all parts to get $-2 \leq -4x \leq 8$, then divide all by -4 to get $1/2 \geq x \geq -2$, so the length of the interval is $1/2-(-2) = 2.50$.

3. What is the sum of the digits of the integer solution to $\sqrt{14 + \sqrt{27 - \sqrt{x-1}}} = 4$?

   (A) 5 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 11

   Solution: C. Square both sides to get $14+\sqrt{27-\sqrt{x-1}} = 16$, then massage it, square again, and solve $27 - \sqrt{x-1} = 4$ to get $x-1 = 529$. Thus $x = 530$ and the sum of the digits is $5+3+0 = 8$.

4. Scott and Adam have some chickens and horses. When asked how many chickens and horses they had altogether, Adam said, "There are a total of 17 heads and 42 feet. I have three times as many chickens as horses." "Yes," replied Scott, "and we both have the same number of horses." How many chickens does Scott have?

   (A) 2 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 13

   Solution: D. Let c and h denote the number of chickens and horses respectively. Then $c+h = 17$ and $2c+4h = 42$ altogether, so $2c+4h-(2c+2h) = 2h = 42-34 = 8$ which means that $h = 4$, and $c = 13$. Since each has 2 horses, Adam has 6 chickens and Scott has the other 7.

5. In a box there are red and blue balls. If you select a handful of them with eyes closed, you have to grab at least 5 of them to make sure at least one of them is red and you have to grab at least 10 of them to make sure both colors appear among the balls selected. How many balls are there in the box?

   (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

   Solution: D. If 5 balls are needed to make sure at least one of them is red, that means that there are 4 blue balls in the box. If you need 10 to make sure both colors appear then there are 9 balls of the more used color –which must be red. Therefore there are $9 + 4 = 13$ balls in the box.

6. Some hikers start on a walk at 9 a.m. and return at 2 p.m. One quarter of the distance walked is uphill, one half is level, and one quarter is downhill. If their speed is 4 miles per hour on level land, 2 miles per hour uphill, and 6 miles per hour downhill, approximately how far did they walk?

   (A) 16.4 miles **(B)** 17.1 miles **(C)** 18.9 miles

   (D) 20.0 miles **(E)** 21.2 miles

   Solution: B. Suppose that they walk $4x$ miles, so that they walk x miles uphill, x miles downhill, and $2x$ level miles. This takes a total time of $\frac{x}{2}+\frac{x}{6}+\frac{2x}{4}$ hours. Setting this equal to 5 hours, we find $x = 30/7$ so that the total distance is $4x = 120/7 \approx 17.1$ miles.

7. Suppose a, b, c are integers such that

   1. $0 < a < b$,

   2. The polynomial $x(x - a)(x - b) - 17$ is divisible by $(x - c)$.

   What is $a + b + c$?

   (A) 14 **(B)** 17 **(C)** 21 **(D)** 24 **(E)** 27

   Solution: C. Since it is divisible by $(x - c)$, we have

   $$c(c - a)(c - b) = 17.$$

   Since $c(c - a)(c - b) = 17 > 0$, it follows that $c > 0$ and we have the following two cases:

Case 1: $0 < (c-b) < (c-a) < c$

Case 2: $(c-b) < (c-a) < 0 < c$.

Since 17 is a prime number, case 1 does not occur. In case 2, $c = 1, c-a = -1, c-b = -17$. Hence $a = 2, b = 18, c = 1$. Thus, $a+b+c = 21$.

8. Let u and v be the solutions to $2x^2 - 3x + c = 0$. If $2uv = 5$, find $u+v+c$.

   (A) -1 **(B)** -1.5 **(C)** 6.5 **(D)** 8.5 **(E)** 10

   Solution: C. Note that $uv = 2.5 = c/2$, so $c = 5$. We have the equation $2x^2 - 3x + 5 = 0$, which we solve to get two solutions, -1 and 2.5. So $u+v+c = -1+2.5+5 = 6.5$.

   Alternatively, since u and v are the solutions, $2x^2 - 3x + c = 2(x-u)(x-v) = 2x^2 - 2(u+v) + 2uv$. Thus $2(u+v) = 3$ and $c = 2uv = 5$. So $u+v+c = 1.5+5 = 6.5$.

9. Let x, y be positive integers with $x > y$. If $1/(x+y) + 1/(x-y) = 1/3$, find $x^2 + y^2$.

   (A) 52 **(B)** 58 **(C)** 65 **(D)** 73 **(E)** 80

   Solution: E. Write $u = x+y$ and $v = x-y$, then u and v are positive integers and $v < u$. Now $1/u + 1/v = 1/3$ so $3 < v < 6$, so $v = 5$ or $v = 4$. If $v = 5$ then $u = 7.5$ which is not an integer. If $v = 4$ then $u = 12$, so we will have $x = 8, y = 4$. Therefore, $x^2 + y^2 = 64 + 16 = 80$.

10. If x and y are integers, under what conditions is $x^2 + xy + (x-y)$ odd?

    (A) x is odd and y is odd **(B)** x is odd and y is even **(C)** x is even and y is odd

    (D) x is even and y is even **(E)** The expression is always even

    Solution: C. A case by case analysis shows that when x is even and y odd, xy is even, x^2 is even, and $x-y$ is odd. Therefore in this case $x^2+xy+(x-y)$ is odd. The other three parity assignments lead to an even integer value.

    Alternatively, no matter whether x is even or odd, $x^2 + x$ is even. So we are left with considering $xy - y = y(x-1)$. The only way for this to be odd is if both factors are odd. So x must be even and y must be odd.

11. In five years Vic will be half as old as his dad. Twenty five years ago he was 1/8 as old as his dad. How old was his dad on the day Vic was born?

    (A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 40

    Solution: D. Let x denote Vic's current age and y Vic's dads current age. The problem information yields the following linear system: $x + 5 = (1/2)(y + 5)$ and $x - 25 = (1/8)(y - 25)$. The system can be solved for $x = 30, y = 65$. So, Vic's dad was $65 - 30 = 35$ on the day Vic was born.

    Alternatively, let x be Vic's current age and y be "Dad's" current age. In making equations, convert "half" to "twice" and "1/8" to "eight times" to get the relations $2(x + 5) = y + 5$ and $8(x - 25) = y - 25$. From these, $2x + 5 = 8x - 175$, so $x = 30$ and $y = 65$. Thus Dad was $35 = 65 - 30$ when Vic was born.

12. A man's salary is reduced by p percent. By what percent would his salary then have to be raised to bring it back to the original amount?

    (A) $2p/(100 - p)$ **(B)** $(p - 100)/(100 - 2p)$ **(C)** $100p/(100 - p)$

    (D) $p/(p - 100)$ **(E)** $2p/(p - 100)$

    Solution: C. Let S denote the man's salary and let x denote the required percent. Then $S - S(p/100) + x/100[S - S(p/100)] = S$. This is equivalent to $S(1-p/100)+S(x/100-xp/10000) = S$. So $1-p/100+x/100-xp/10000 = 1$. Solving for x, we get $x = 100p/(100 - p)$.

13. Benny eats a box of cereal in 14 days. He eats the same size box of cereal with his younger brother Nathan in 10 days. How many days will it take Nathan to finish the box of cereal alone?

    (A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 40

    Solution: D. Within $10 \times 14 = 140$ days Benny will eat 10 boxes of cereal alone, while together with Nathan they will eat $140 \div 10 = 14$ boxes for the same time period. That means the share of his brother is $14 - 10 = 4$ boxes for 140 days. Therefore, Nathan eats one box of cereal for $140 : 4 = 35$ days.

14. Given the following system of equations

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{1}{3} \\ \frac{1}{x}+\frac{1}{z} &= \frac{1}{5} \\ \frac{1}{y}+\frac{1}{z} &= \frac{1}{7} \end{aligned}$$

What is the value of the ratio $\frac{z}{y}$?

(A) 17 **(B)** 23 **(C)** 29 **(D)** 31 **(E)** 36

Solution: C. Add the three equations together to get $(*)\quad 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{3}+\frac{1}{5}+\frac{1}{7} = \frac{35+21+15}{105} = \frac{71}{105}$. Then subtract the $2\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{2}{3}$ from both sides to get $\frac{2}{z} = \frac{71}{105} - \frac{70}{105} = \frac{1}{105}$, so $z = 210$. Subtracting $2\left(\frac{1}{x}+\frac{1}{z}\right) = \frac{2}{5}$ from both sides of $(*)$ yields $\frac{2}{y} = \frac{71}{105} - \frac{42}{105} = \frac{29}{105}$, so $y = \frac{210}{29}$. It follows that $\frac{z}{y} = \frac{210}{\frac{210}{29}} = 29$.

Alternatively, change variables to $u = 1/x$, $v = 1/y$ and $w = 1/z$. Then (using the new forms) subtract the second equation from the first to get $v-w = 2/15$. Add this to the third to get $v = 29/210$. Substitute into the third to get $w = 1/210$. Thus $z/y = v/w = 29$.

15. A two-digit number N is 10 more than 3 times the sum of its digits. The units digit is 1 more than twice the 10's digit. Find the product of the digits.

(A) 24 **(B)** 28 **(C)** 30 **(D)** 32 **(E)** 36

Solution: E. Let x and y denote the tens and units position of N. Then $N = 10x + y$, $10x + y = 10 + 3(x + y)$, and $y = 2x + 1$. Replacing y with $2x + 1$ yields $10x + 2x + 1 = 10 + 9x + 3$ which reduces to $3x = 12$, whence $x = 4$ and $y = 9$.

Alternatively, since the units digit, y, is at most 9 and is 1 more than twice the tens digit, x, the tens digit must be less than or equal to 4. For $N = 10x + y$, the possibilities are 13, 25, 37 and 49. All are 10 more than a multiple of 3, so next check the first condition: $3\cdot(1+3) = 12 \neq 13-10$, $3\cdot(2+5) = 21 \neq 25-10$, $3\cdot(3+7) = 30 \neq 37-10$ and $3\cdot(4+9) = 39 = 49-10$. Thus $N = 49$, and the product of its digits is 36.

16. Let A, B, and C be digits satisfying

$$\begin{array}{r} A\ B \\ +\ A\ A \\ \hline C\ B\ 2 \end{array}$$

What is $A+B+C$?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Solution: D. Clearly $C=1$. Expressing the problem in decimal notation, we have $20A+A+B=100+10B+2$ which is equivalent to $21A-9B=102$. It follows that $A \geq 5$. Checking values of A, we find that the only possible value of A for which $102-21A$ is a multiple of 9 is $A=7$, It follows that $B=5$, so the sum is $7+5+1=13$.

17. For how many integer values of n is

$$\frac{3}{17} < \frac{n}{68} < \frac{32}{51}?$$

(A) 28 **(B)** 29 **(C)** 30 **(D)** 32 **(E)** 34

Solution: C. For positive numbers a, b, c, d, $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$, it follows that $3 \cdot 17 \cdot 4 < 17 \cdot n$ and $3 \cdot 17 \cdot n < 32 \cdot 4 \cdot 17$. Solving these simultaneously for n yields $13 \leq n \leq 42$ which is 30 values.

18. A book has pages numbered starting at 1. The digit 3 is printed 237 times during page number printing. What is the greatest number of pages the book could have?

(A) 663 **(B)** 664 **(C)** 672 **(D)** 673 **(E)** 683

Solution: C. Each decade of numbers $0, 1, 2, 3, \ldots, 99$; $100, 102, \ldots, 199$; $\ldots$ $900, 902, \ldots 999$, except the $300-399$ decade has exactly 20-3s. The number of 3 printed on the pages 300 to 399 is 120. Adding these groups of 20, we see that a 700 page book would have 240 copies of the digit 3 printed. Removing the last three 3s means the book could not have the pages 693, 683, or 673. So the largest number of pages it could have is 672.

19. The odd numbers from 1 to 17 can be used to build a 3×3 magic square (the rows and columns have the same sum). If the $1, 5$, and 13 are as shown, what is x?

(A) 7 **(B)** 9 **(C)** 11 **(D)** 15 **(E)** 17

	1	
5		13
x		

Solution: A. The magic sum is 27 because the sum of the first nine odd positive integers is $9^2 = 81$. This means that we can fill in the 9 and the 17 as shown.

	1	
5	9	13
x	17	

We can quickly eliminate $x = 11$ and $x = 15$. If $x = 3$, then we would have to use the 7 in the bottom right square, which would require another 7 in the top right square. Thus $x = 7$. The complete square is shown below.

15	1	11
5	9	13
7	17	3

20. Let a and b be two different real numbers. The equation $(x-a)(x-b-1)+(x-a)(x-b+1)=0$

(A) has three roots, one of which is $x = b$ **(B)** has three roots, one of which is $x = a$

(C) has three roots, one of which is $x = b+1$ **(D)** has two roots, one of which is $x = b$

(E) has two roots, one of which is $x = b+1$

Solution: D. Factor and solve to get $(x-a)((x-b-1)+(x-b+1)) = (x-a)(2x-2b) = 0$, so the two roots $x = a$ and $x = b$

21. Let N denote the two-digit number whose cube root is the square root of the sum of its digits. How many positive divisors does N have?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

    Solution: C. There are just two two-digit cubes, 27 and 64. Checking each one shows that 27 satisfies the requirements and 64 does not. The number 27 has 4 divisors.

22. Find the number of odd divisors of 7!.

    (A) 4 **(B)** 6 **(C)** 10 **(D)** 12 **(E)** 24

    Solution: D. The prime factorization of 7! is $2^4 \cdot 3^2 \cdot 5 \cdot 7$. Each odd factor of 7! is a product of odd prime factors, and there are $3 \cdot 2 \cdot 2 = 12$ ways to choose the three exponents.

23. Farmer Brown's chickens are unusual. One and a half of them can lay an egg and a half in a day and a half, on average. At this rate, how many days would it take all 60 of his chickens to lay 480 eggs?

    (A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 20

    Solution: C. $3/2$ chickens $\times 3/2$ days $= 3/2$ eggs implies that the rate of egg laying is $2/3$ eggs per chicken per day. Letting d denote the number of days required, we have $60 \times 2/3 \times d = 480$ and it follows from this that $d = 12$.

    Alternatively, doubling the number of chickens, but not the number of days, will give 3 chickens producing 3 eggs in one and a half days. Doubling the time to 3 days, the 3 chickens will produce 6 eggs in 3 days. So $60 = 20 \cdot 3$ chickens will produce $20 \cdot 6 = 120$ eggs in 3 days. Thus it will take $12 = 4 \cdot 3$ days for the 60 chickens to produce 480 eggs.

    Yet another alternate solution: Since $60 = 40 \cdot 3/2$, all 60 chickens will produce $40 \cdot 3/2 = 60$ eggs in one and a half days. This is one–eighth of the required total, so $8 \cdot 3/2 = 12$ is the how long it will take to have 480 eggs.

24. The radius of the circle given by

$$x^2 - 6x + y^2 + 4y = 12$$

is

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 36

Solution: A. Complete the squares by adding 9 and 4 to both sides to get

$$x^2 - 6x + 9 + y^2 + 4y + 4 = (x-3)^2 + (y+2)^2 = 12 + 9 + 4 = 25 = 5^2.$$

So the radius is 5.

Alternate solution: Solve $x^2 - 6x = 0$ and $y^2 + 4y = 0$. Obviously the solutions are $x = 0, 6$ and $y = 0, -4$. From this it follows that the center of the circle is the point (h, k) where $h = (0+6)/2 = 3$ and $k = (0+(-4))/2 = -2$. One way to finish: substitute $x = 3$ (or $y = -2$) into the equation $x^2 - 6x + y^2 + 4y = 12$ and solve for y (or x). Have $0 = y^2 + 4y - 21 = (y+7)(y-3)$, so $(3, -7)$ and $(3, 3)$ are on the vertical diameter. Thus the diameter is 10 and the radius is 5. OR: use that the standard form for the equation is $(x-3)^2 + (y+2)^2 = r^2$ and from this deduce that $r^2 = 9 + 4 + 12 = 25$.

25. The integer-valued numerator of a certain fraction is 10 more than the denominator. If the number that is two larger than the denominator is doubled, the result is the same as the numerator. In which interval does the value of the fraction belong?

(A) $(0, 1]$ **(B)** $(1, 2]$ **(C)** $(2, 3]$ **(D)** $(3, 4]$ **(E)** $(4, 5]$

Solution: C. Let d and n denote the denominator and numerator respectively. Then d and n satisfy $n = d + 10$ and $2(d+2) = n$ from which it follows that $d + 10 = 2(d+2)$, whence $d = 6$ and $n = 16$. So the fraction is $\frac{16}{6} = \frac{8}{3}$.

26. John was contracted to work A days. For each of these A days that John actually worked, he received B dollars. For each of these A days that John didn't work, he had to pay a penalty of C dollars. After the A days of contracted work was over, John received a net amount of D dollars for his work. How many of the A days of contracted work did John not work?

    (A) $(AB - D)/(B + C)$ **(B)** $(AB + D)/(B + C)$ **(C)** $(AB - D)/(B - C)$
    (D) $(AB + D)/(B - C)$ **(E)** $(AC - B)/(D - C)$

    Solution: A. Let x denote the number of days John did not work. Then he worked $A - x$ days and so earned $B(A - x) - Cx = D$ dollars. Solving this for x, we get $-(B + C)x = D - AB$ and so $x = (AB - D)/(B + C)$. Taken from Horatio Nelson Robinson's 1859 book 'A Theoretical and Practical Treatise on Algebra', with thanks to Dave Renfro.

27. Let N denote the 180-digit number obtained by listing the 90 two-digit numbers from 10 to 99 in order. Thus $N = 10111213\ldots99$. What is the remainder when N is divided by 99?

    (A) 0 **(B)** 10 **(C)** 45 **(D)** 54 **(E)** 90

    Solution: D. Let U denote the sum of the 90 units digits and T the sum of the 90 tens digits. Thus $U = 0 + 1 + 2 + \cdots + 9 + \cdots + 9 = 405$ and $T = 1 + 1 + \cdots + 1 + 2 + 2 + \cdots + 9 + 9 + \cdots + 9 = 450 = U + 45$. Since $99 = 9 \cdot 11$, we ask the two questions 'What is the remainder when N is divided by 11' and 'What is the remainder when N is divided by 9'. The divisibility test for 11 tells us that the remainder when a number N is divided by 11 is the same as when the alternating sum of the digits of N is divided by 11. That alternating sum is $U - T = -45$, and the remainder when $-45 = -5(11) + 10$ is divided by 11 is 10. The remainder when N is divided by 9 is the same as the remainder when the sum of digits of N is divided by 9. This sum of digits is $U + T = 9 \cdot 45 + 9 \cdot 50 = 9 \cdot 95$ which is a multiple of 9. So far we know that there are integers p and q such that $N = 9p$ (N is a multiple of 9) and $N = 11q + 10$ (when N is divided by 11, the remainder is 10). Let r be the remainder when N is divided by 99. Then there is an integer k such that $N = 99k + r$. Now $N/9 = 11k + r/9 = p$ shows r is a multiple of 9. Furthermore $N/11 = 9k + r/11 = q + 10/11$ shows $r = 11(q - 9k) + 10$ shows r is 10 more than a multiple of 11. The only number in the range 0 to 98 that is both a multiple of 9 and 10 bigger than a multiple of 11 is 54.