1. Let a, b be the two solutions to the equation $x^2 - 3x + 1 = 0$. Find $a^3 + b^3$.

   (A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 24

   (D) The sum of the roots of $ax^2 + bx + c = 0$ is $-b/a$ and the product is c/a. Therefore $a + b = 3$ and $ab = 1$, and it follows that

   $$a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)[(a + b)^2 - 3ab] = 18.$$

2. A man has a pocket full of change, but cannot make change for a dollar. What is the greatest value of coins he could have?

   (A) \$.99 **(B)** \$1.09 **(C)** \$1.19 **(D)** \$1.29 **(E)** \$1.39

   (C) The man could have three quarters, four dimes, and four pennies, for a total of \$1.19.

3. For what value(s) of b does the equation $2x^2 + bx + 2 = 0$ have no real roots?

   (A) $b > 2$ **(B)** $-4 < b < 4$ **(C)** $b > 2$ or $b < -2$

   (D) $b < -4$ **(E)** $b = 4$

   (B) Since the equation is quadratic, the solutions are real if and only if $b^2 - 4 \cdot a \cdot c \geq 0$, or $b^2 - 16 \geq 0$. The solution is $b \geq 4$ or $b \leq -4$. Hence, the equation will have no real roots when $4 < b < 4$.

4. A boy is 4 feet 8 inches tall. If he could walk around the earth one time on the equator, the top of his head would travel farther than his feet. Assuming that the equator is a perfect circle, how much farther would the top of his head travel than his feet (in inches)?

   (A) 56 **(B)** 56π **(C)** 96π **(D)** 112π **(E)** 3136π

   (D) Since we assume the equator to be a circle, then the distance the boys feet and head travel are given by $C = 2\pi r$ and $C = 2\pi(r + 56)$ respectively, where r is the radius of the earth. We then expand the second quantity to get $C = 2\pi(r + 56) = 2\pi r + 112\pi$. Hence, the boys feet travel 112π inches more than his head.

5. A line L has a slope of -2 and passes through the point $(r,-3)$. A second line, K is perpendicular to L at (a,b) and passes through the point $(6,r)$. The value of a is

   (A) r **(B)** $2r/5$ **(C)** 1 **(D)** $2r-3$ **(E)** $5r/2$

   (B) An equation for L is $y+3=-(x-r)$, so $y=-2x+(2r-3)$. An equation for K is $y-r=1/2(x-6)$, so $y=1/2x+(r-3)$. Thus $-2x+2r-3=1/2x+r-3$, so $x=2r/5$.

6. Which of the following describes the range of the function $f(x)=x^2 2x$ if its domain is the three element set $\{0,2,4\}$?

   (A) $\{0\}$ **(B)** $\{0,2\}$ **(C)** $\{0,8\}$

   (D) $\{2,8\}$ **(E)** all real numbers equal to or greater than -1

   (C) Note that $f(0)=0$, $f(2)=0$ and $f(4)=8$, so the range is $\{0,8\}$.

7. The slope of a line perpendicular to the line passing through the points $(1,3)$ and $(-2,1)$ is

   (A) $-3/2$ **(B)** $-2/3$ **(C)** 0 **(D)** $2/3$ **(E)** $3/2$

   (A) The slope of the given line is $(1-3)/(-2-1)=2/3$. The slope of the line perpendicular to the given line is the negative reciprocal, $-3/2$.

8. The numerator of a certain fraction is 5 less than the denominator. If 3 is subtracted from the numerator and 2 is added to the denominator, the resulting fraction is equal to one half. What is the original fraction?

   (A) 9/18 **(B)** 13/18 **(C)** 9/13 **(D)** 17/18 **(E)** 4/18

   (B) Letting D be the denominator, the numerator of the new fraction is $D-5-3$ and the denominator of the new fraction is $D+2$. Solving $(D-5-3)/(D+2)=1/2$ gives 13 for the numerator and $D=18$ for the denominator.

9. Niki runs at a rate of 800 ft. per minute. Jeff runs at the rate of 900 ft. per minute. If Niki starts 500 feet ahead of Jeff, how many minutes does it take Jeff to catch up with her.

   (A) 1.5 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 7

   (D) Jeff gains 100 feet per minute, so it takes him 5 minutes to catch up.

10. A recent poll showed that nearly 30% of European school children think that

$$\frac{1}{2}+\frac{1}{3}=\frac{2}{5}.$$

This is wrong, of course. Is it possible that $\frac{1}{a}+\frac{1}{b}=\frac{2}{a+b}$ for some real numbers a and b?

(A) Yes, and $a+b=1$ **(B)** Yes, and $a+b=2$ **(C)** Yes, and $a^2+b^2=1$
(D) Yes, and $a^2+b^2=0$ **(E)** No, it is not possible

(E) The given equation is equivalent to $\frac{a+b}{ab}=\frac{2}{a+b}$, which is true only if $a^2+2ab+b^2=2ab$, but this can happen only when $a^2+b^2=0$, which implies that $a=b=0$, in which case the equation in nonsense.

11. What is the number of real solutions of the equation

$$\frac{x^6-8}{x^2-2}=12?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

(A) Introducing $y=x^2$ we may rewrite or equation as

$$\frac{y^3-8}{y-2}=12. \tag{1}$$

Using $y^3-8=y^3-2^3=(y-2)(y^2+2y+4)$, we may simplify our equation to $y^2+2y+4=12$, or $y^2+2y-8=0$. Factoring or the quadratic formula yields $y=2$ or $y=-4$. Here $y=-4$ must be discarded, since the square of x can not be negative. On the other hand $y=2$ is not a solution of the intermediary equation (1), since substituting $y=2$ yields $\frac{0}{0}$ on the left hand side. Therefore the equation has no real solutions.

12. What is the equation of the tangent line of the circle $x^2+y^2=2$ at the point $(1,1)$?

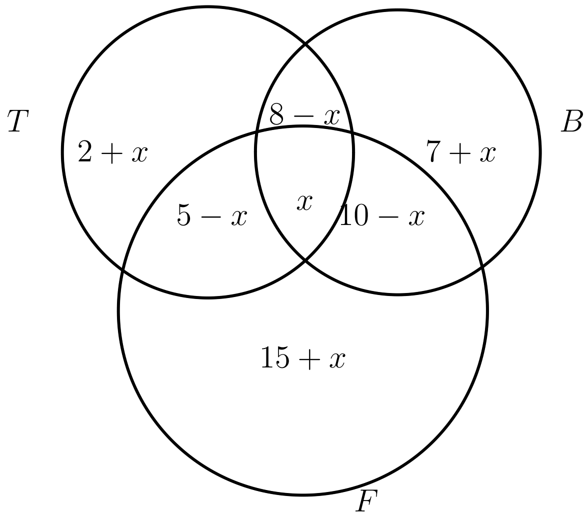
(A) $x=1$ **(B)** $y=1$ **(C)** $x+y=2$ **(D)** $x-y=0$ **(E)** $x+y=0$

(C) The center of our circle is $(0,0)$ and the line connecting the circle with $(1,1)$ has slope 1. Thus the tangent line must have slope -1 and pass through $(1,1)$. The point-slope form of the equation of the tangent line is $y-1=-(x-1)$, which may be rearranged into $x+y=2$.

13. Conard High School has 50 students who play on the baseball, football, and tennis teams. Some students play more than one sport. If 15 play tennis, 25 play baseball, 30 play football, 8 play tennis and baseball, 5 play tennis and football, and 10 play baseball and football, determine how many students play all three sports.

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

    (B) Let TBF represent the number of students that play all three sports, let B be the number of students that play baseball, let F be the number of students that play football, and let T be the number of students that play tennis. Let TB, BF, and TF represent the number of students who play the appropriate combinations of the individual sports. Then a Venn diagram shows that the total number of students (ie, 50) equals $T + B + F - TB - TF - BF + TBF$. Therefore $50 = 15+25+30-8-5-10+TBF$ which implies that $TBF = 3$. Alternatively, label each region of the Venn diagram as shown.



14. Suppose N is a positive integer that is a perfect cube. Which of the following represents the next positive integer that is a perfect cube?

    (A) $N^3 + 3\sqrt[3]{N} + 1$ **(B)** $N + 3\sqrt[3]{N^2} + 3\sqrt[3]{N} + 1$ **(C)** $N^3 + 3N^2 + 3N + 1$
    (D) $N^3 + N^2 + N + 1$ **(E)** N^3

    (B) If $N = x^3$, then $(x+1)^3 = x^3 + 3x^2 + 3x + 1 = N + 3\sqrt[3]{N^2} + 3\sqrt[3]{N} + 1$.

15. How many ordered pairs of integers with a sum 23 have a product that is maximal?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

    (C) The product $12 \cdot 11 = 132$ is the largest product possible among pairs of integers with sum 23, and there are two ordered pairs $(12, 11)$ and $(11, 12)$ whose product is 132.

16. The sum of the zeros of $f(x) = x(2x + 3)(4x + 5) + (6x + 7)(8x + 9)$ is

    (A) $\dfrac{-35}{4}$ **(B)** $\dfrac{35}{4}$ **(C)** -70 **(D)** 70 **(E)** 0

    (A) Since $f(x) = 8x^3 + 70x^2 + 125x + 63$, the sum of zeros is $\frac{-70}{8} = \frac{-35}{4}$.

17. Three points $A = (0, 1)$, $B = (2, a)$, and $C = (3, 7)$ are on the straight line. What is the value of a ?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    (B) The slope of $\overline{AB}$ is the same as that of $\overline{AC}$. That is $\frac{a-1}{2-0} = \frac{7-1}{3-0}$.

18. If $x > 0$ and if $5, 6 + x, 7 + x$ are the lengths of a right triangle, what is x ?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

    (A) Since $7 + x$ is largest, $5^2 + (6 + x)^2 = (7 + x)^2$ so that $2x - 12 = 0$ and $x = 6$.

19. One leg of a right triangle is two meters longer than twice the length of the other leg. The hypotenuse is four meters less than the sum of the two legs. What is the perimeter of the triangle, in meters?

    (A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 50

    (C) The three sides have lengths x, $2x+2$, and $3x-2$, and these numbers satisfy the Pythagorean identity $a^2+b^2 = c^2$. Thus, $x^2+4(x^2+2x+1) = 9x^2-12x+4$, which is equivalent to $4x^2 - 20x = 0$, so the solutions are $x = 0$ and $x = 5$. Therefore the perimeter is $5 + 12 + 13 = 30$.

20. Margaret and Cyprian both have some nickels, dimes and quarters, at least one of each type and a different number of each type. Margaret has the same number of quarters as Cyprian has dimes, and she has the same number of dimes as Cyprian has nickels. She also has the same number of nickels as Cyprian has quarters. The value of their coins is the same. What is the least value they could have?

    (A) 85 **(B)** 95 **(C)** 105 **(D)** 125 **(E)** 135

    (A) Let K denote this least value and let n, d, and q denote the number of nickels, dimes, and quarters respectively Margaret has. Then $5n+10d+25q = K$ and $25n + 10q + 5d = K$. Subtracting one equation from the other yields $d = 4n - 3q$. We want to minimize q, so we try $q = 1$. Then $n = 1$ is not allowed so we try $n = 2$ and we get $d = 5$, so the value of K is 85. The other possible values can be easily eliminated.

21. For what value of k are the lines $2x+3y = 4k$ and $x-2ky = 7$ perpendicular?

    (A) $-3/4$ **(B)** $1/6$ **(C)** $1/3$ **(D)** $1/2$ **(E)** $2/3$

    (C) The slope of the first line is $-2/3$ so the slope of the second, which is $1/2k$ must be $3/2$. Hence, $k = 1/3$.

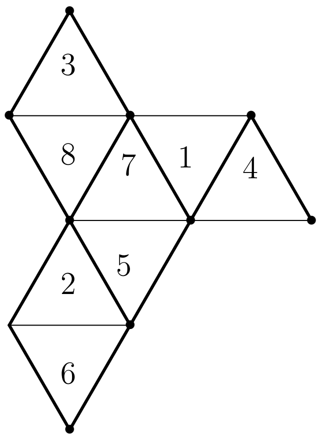
22. Suppose $\{a, b, c, d, e, f\} = \{2, 3, 4, 5, 6, 7\}$. What is the least possible value of $ab + cd + ef$?

    (A) 50 **(B)** 52 **(C)** 53 **(D)** 60 **(E)** 68

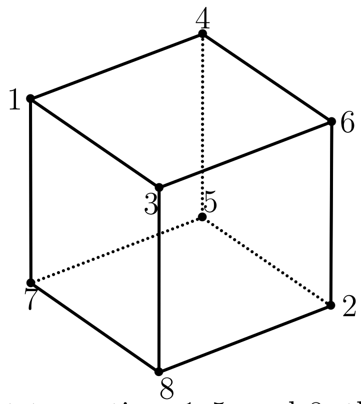
    (B) The smallest value is the one obtained by pairing up the largest and smallest members of the set. Thus $2 \cdot 7 + 3 \cdot 6 + 4 \cdot 5 = 52$ is the least sum obtainable.

23. An octahedral net is a collection of adjoining triangles that can be folded into a regular octahedron. When the net below is folded to form an octahedron, what is the sum of the faces adjacent to one marked with a 3?

    (A) 13 **(B)** 15 **(C)** 17 **(D)** 18 **(E)** 19



(B) The octagon is dual to the cube. In other words, if we make each face of the octagon into a vertex and declare two faces (vertices) to be adjacent if the faces have an edge in common, the resulting figure is a cube.



Note that the vertex 7 is adjacent to vertices $1, 5$, and 8, that 8 must be adjacent to both 3 and 2. Continuing in this way, find that 3 is adjacent to $1, 6$, and 8 for a total of 15.

24. Let $f(a,b,c) = \frac{a+b}{c} + \frac{b+c}{a} + \frac{a+c}{b}$. There is an integer u such that $f(u-1, u, 2u) = 8$. What is $f(u^2-3, 2u-3, 2u-4)$?

    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

    (E) First solve $\frac{2u-1}{2u} + \frac{3u-1}{u} + \frac{3u}{u-1} = 8$. The equation is equivalent to $14u^2 - 11u + 3 = 8(2u)(u-1)$ which leads to $u = 3$. Thus we need to compute $f(6,3,2)$ which is 8.

25. Let M and N denote the two integers that are respectively twice and three times the sum of their digits. What is $M + N$?

    (A) 27 **(B)** 36 **(C)** 45 **(D)** 54 **(E)** 60

    (C) Let $M = 10a + b$. Then $2(a+b) = 10a + b$ and it follows that $8a = b$ and $M = 18$. Similarly, we can show that $M = 27$, so their sum is 45.

26. How many integer solutions (x, y, z) does $\frac{1}{x} + \frac{y}{z} = \frac{13}{21}, x < 7$ have, where y and z are relatively prime?

    (A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 7

    (C) This solution is inadequate. I'll expand it if it makes the cut. Just list the answer, using the "formula" $\frac{y}{z} = \frac{13}{21} - \frac{1}{x}$. The five solutions are $(x, y, z) = (2, 5, 42); (3, 2, 7); (4, 31, 84); (5, 44, 105)$; and $(6, 19, 42)$.