My Favorite Problems, 4
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This is the fourth of a series of columns about problems. I am soliciting problems from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions usually require a clever use of a well-known problem solving technique.

4.1 This problem came to me from Stan Wagon, Macalester College, who credits Frank Rubin. Suppose you have n keys on a circular key chain and wish to put a colored sleeve on each key so that it is identifiable by its color alone, without reference to its shape. For some n this can be done with fewer than n colors. For example, if you have 4 keys then you can place the colors on the keys as follows:

Red

Red *Green*

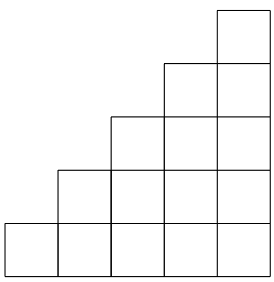
Blue

The top key is the red key that is across from the blue key; and the leftmost key is the red key that is adjacent to a blue one; the other two are identifiable by their colors.

An identification scheme must work even if the key chain is flipped over, so one cannot use the words right and left. Let $f(n)$ be the least number of colors necessary so that each key on a chain of n keys can be uniquely identified as explained above. Then $f(1) = 1, f(2) = 2$, and $f(3) = 3$. What is $f(123)$?

4.2 For each ordered pair (j, k) of nonnegative integers for which the number 5^j7^k is less that 10^{29}, there are three consecutive positive integers (perhaps four) $i, i+1$, and $i+2$ such that $10^{29} \le 2^i5^j7^k < 2^{i+1}5^j7^k < 2^{i+2}5^j7^k < 10^{30}$. There are more than one thousand 30-digit numbers of the form $2^i5^j7^k$. Prove that in every 30-digit number of this form, some digit appears at least 4 times in its decimal representation.

4.3 An n-staircase is a grid of $1 + 2 + \cdots + n = \binom{n+1}{2}$ squares arranged so that column 1 has 1 square, column 2 has 2 squares, ..., and column n has n squares. My friend Rick Armstrong, St Louis Community College, posed in Mathematics and Computer Education, (`http://www.macejournal.org/`) the question of how many square regions are bounded by the gridline of an n-staircase. My question for readers if this: how many rectangular regions are bounded by the gridlines of an n staircase? The figure shows a 5-staircase.



Solutions to problems in Favorite Problems 3.

3.1 Linked Triangles. This beautiful problem is due to Sam Vandervelde, Greater Testing Concepts. Let A, B, C, D, E, and F be six points in 3-space, no four of which lie on the same plane. (I.e. six points in general position.) We say that triangles ABC and DEF are "linked" if exactly one of segments AB, AC, or BC intersects the interior of triangle DEF. This has the effect of linking the triangles, so that if the sides were made of thin metal rods it would be impossible to separate them. (Try drawing a picture!) The problem is to show that no matter how the six points are positioned in space (as long as no four are coplanar) it is always possible to split them into two sets of three points each so that the two resulting triangles (with the points in each set as vertices) are linked.

Solution: PROOF: The boundary of the convex hull of the given six points contains either four, five, or six points since at least four vertices are needed to create a solid. We will consider each of these cases in turn, but first note that in every case the faces of the outer polyhedron P are triangles, because a face of the outer polyhedron is contained in a single plane. A face with four or more vertices would imply that these four or more points were coplanar, contradicting the hypothesis in the setup section.

Case 1: We begin by assuming that the convex hull consists of all six points, which means the outer polyhedron has six vertices. For the outer polyhedron denote the number of vertices by v, edges by e, and faces by f. If we count the number of edges around all the faces we obtain $3f$, since all faces are triangular. Each edge is included twice in this count, since each edge is part of exactly two faces, thus $e = 3f/2$. We have $v = 6$, so by Euler's formula we find

$$v + f = e + 2 \quad \Longrightarrow \quad 6 + f = \frac{3f}{2} + 2 \quad \Longrightarrow \quad f = 8,$$

and therefore $e = 3f/2 = 12$.

Sam credits Naperville North High School for the strategy for the next segment of the proof. His original proof has been modified here. Let $v_A, v_B, v_C, \ldots, v_F$ denote the degree of each of the vertices $A, B, C, \ldots, F$ of P respectively. That is, the number of edges of P that have the vertex as an endpoint. Then

$$v_a + v_B + \cdots + v_F = 2e = 24$$

because there are 12 edges and each one has two endpoints. We claim that $v_x = 4$ for some vertex x. To prove this note that for each vertex x, $3 \leq v_x \leq 5$ since there are only five other vertices to which x could be connected, and since at least three edges must meet at x to form a solid angle. If there are no vertices of degree four, then there must be three of degree three and three others of degree five. Suppose $v_A = v_B = v_C = 5$. Then at least two members of the set $\{D, E, F\}$ are adjacent. Suppose (as in Fig 1) it is D and E. Then D is also adjacent to A, B, and C, contradicting $v_D = 3$.

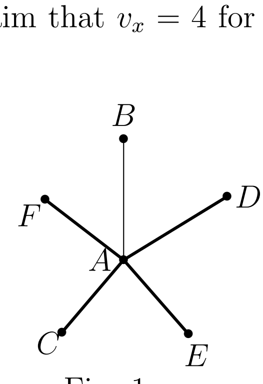


Fig. 1

Hence some vertex lies on 4 edges, say A. Without loss of generality, assume A is connected to B, C, D, and E. Since $v_F \geq 3$ and F is not joined to A it must be connected to at least three of B, C, D, and E; thus to either B and D or C and E by the Pigeonhole Principle.

Now we will explicitly use the convexity of P. Specifically, if $\overline{AF}$ is not an edge of the polyhedron then it must lie in the interior. Therefore the part of segment AF near A lies inside the solid angle at A which is composed of the tips of the two tetrahedra $ACEB$ and $ACED$. The segment can't lie along plane ACE or else the four points A, C, E, and F would be coplanar; so it lies inside one of these two tetrahedra, say $ACED$. Finally, $\overline{AF}$ must extend beyond the base of this tetrahedron or else F would be in the interior of the polyhedron. See Fig. 2. Therefore $\overline{AF}$ intersects the interior of $\triangle CDE$. We now claim that triangles ABF and CDE are linked. We have just shown that AF intersects $\triangle CDE$, and as AB and FB are edges of the polyhedron they cannot possibly intersect $\triangle CDE$, so by definition we have found a pair of linked triangles. The other possibility is that $\overline{AF}$ lies inside tetrahedron $ACEB$, in which case we would find analogously that triangles ADF and CBE were linked.

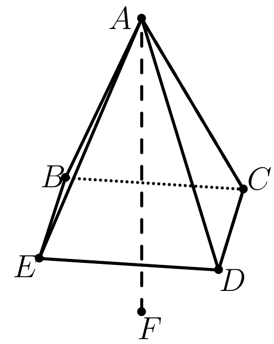


Fig. 2

Case 2: In this case the convex hull consists of five points, so P has five vertices with the sixth point in its interior. Denote the number of vertices by v, edges by e, and faces by f. If we add together the number of edges around each face we obtain $3f$, since each face has three sides. Every edge is included twice in this count, since each edge is part of exactly two faces. Combining these observations yields $e = 3f/2$. We are given that $v = 5$, so by Euler's formula we can deduce $v + f = e + 2 \implies 5 + f = \frac{3f}{2} + 2 \implies f = 6$. It now follows that $e = 3f/2 = 9$.

Note that every vertex must be joined to at least three other vertices by edges of P in order to form a three dimensional solid angle. Of course, no vertex is the endpoint of more than four edges since there are only four remaining vertices on P. If every vertex were the endpoint of four edges we would have a total of 20 endpoints, or 10 edges altogether. This contradicts the fact that $e = 9$, so at some vertex only three edges meet.

We assume that A is the vertex at which only three edges meet, say edges $\overline{AB}, \overline{AC}$, and $\overline{AD}$. See Fig. 3. Therefore $\overline{AE}$ is a segment between two vertices of P which is not an edge. Since P is convex this segment must lie in the interior. We just noted that every vertex is the endpoint of at least three edges, and since $\overline{AE}$ is not an edge E must connect to all of B, C, and D.

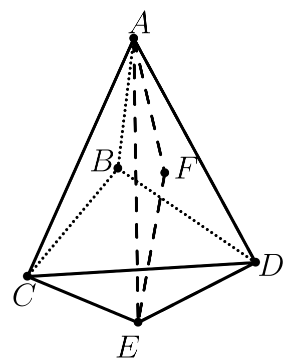


Fig. 3

We can now deduce the general configuration of the five vertices. There are three edges ending at each of A and E. Since $e = 9$ there are three edges remaining, which must be $\overline{BC}, \overline{BD}$, and $\overline{CD}$. Clearly A and E cannot both be on the same side of the plane through $\triangle BCD$, for if they were either AE would be an edge of P or one of A and E would be within P, contradicting what we know. Therefore P must appear as in Fig. 4.

The part of segment AE near A lies inside the solid angle at A which is composed of the tips of the three tetrahedra $ABCF$, $ABDF$, and $ACDF$. The segment can't lie along one of the faces of these tetrahedra, such as $\triangle ACF$; or else four points would be coplanar, in this case the four points A, C, E, and F. Therefore it lies inside one of these three tetrahedra, say $ABDF$. Finally, $\overline{AE}$ must extend beyond the base of this tetrahedron or else E would be in the interior of the polyhedron. Therefore $\overline{AE}$ intersects the interior of $\triangle BDF$. We now claim that triangles ACE and BDF are linked. We have just shown that AE intersects $\triangle BDF$, and as AC and CE are edges of P they cannot possibly intersect $\triangle BDF$, so by definition we have found a pair of linked triangles.

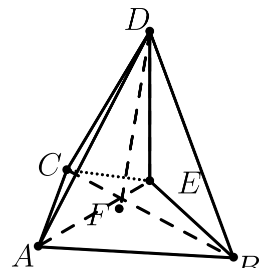


Fig. 4

Case 3: The techniques employed in this case are reminiscent of those above, so we will streamline the argument. Since the convex hull contains only four points P must be a tetrahedron. We will assume that points E and F are in the interior of the tetrahedron. Imagine drawing segments $\overline{EA}, \overline{EB}, \overline{EC}$, and $\overline{ED}$ which subdivide the outer tetrahedron into four smaller tetrahedra. As before, point F cannot lie on any of the faces of these tetrahedra since no four points are coplanar. Suppose that F lies in the interior of $ABCE$. In the same manner as the previous problem we see that $\overline{DF}$ must intersect the interiors of one of the triangles ABE, ACE, or BCE; suppose it intersects $\triangle ACE$. We claim that triangles ACE and BDF are linked. We have just seen that $\overline{DF}$ intersects $\triangle ACE$. Segment BD cannot intersect $\triangle ACE$ since $\overline{BD}$ is an edge of the outer tetrahedron, and segment BF likewise cannot intersect $\triangle ACE$ as BF lies in the interior of $ABCE$ while $\triangle ACE$ is a face of this tetrahedron. Notice that the arguments above are perfectly general; no matter in which tetrahedron F was located, or what face of that tetrahedron was intersected by the segment joining F to the opposite vertex, the same arguments would produce a pair of linked triangles. Only the letters would change.

3.2 Version A. A teacher whispers a positive integer p to student P, q to student Q, and r to student R. The students don't know one another's numbers but they know the sum of the three numbers is 14. The students make the following statements:

(a) P says 'I know that Q and R have different numbers'.

(b) Q says 'I already knew that all three of our numbers are different'.

(c) R says 'Now I know all three of our numbers'.

What is the product of the three numbers?

Version B. A teacher whispers a positive integer p to student P, q to student Q, and r to student R. The students don't know one another's numbers but they know the sum of the three numbers is 14. The students make the following statements:

(a) P says 'I know that Q and R have different numbers'.

(b) Q says '<u>Now I know</u> that all three of our numbers are different'.

(c) R says 'Now I know all three of our numbers'.

What is the product of the three numbers?

Solution: Version A. P's statement implies that p is odd. Q's statement implies that q is odd and also that $q \geq 7$. There are just six triplets (p, q, r) that satisfy all three statements (a) $p+q+r = 14$, (b) p is odd and (c) $q \geq 7$ and odd. They are $(1, 7, 6), (1, 9, 4), (1, 11, 2), (3, 7, 4), (3, 9, 2), (5, 7, 2)$. Since student R reasons perfectly, he can establish that only the six triples listed above are possible. In order to make his statement, r must be 6. Thus the product is $1 \times 7 \times 6 = 42$.

Version B. P's statement implies that p is odd. Q's statement implies that q cannot be 1, 3, or 5 because he could not know that his number is different from P's. If $q = 4, 8$ or 12, Q could not know that p is different from r, because we could have $p = r = 5, p = r = 3$, and $p = r = 1$ respectively. Also, q is not $7, 9$ or 11 because if it were, Q would have known already that all three numbers were different. Thus $q = 2, 6$, or 10. There are just 10 triplets (p, q, r) that satisfy all three statements (a) $p + q + r = 14$, (b) p is odd and (c) $q \in \{2, 6, 10\}$. They are $(1, 2, 11), (1, 6, 7), (1, 10, 3), (3, 2, 9), (3, 6, 5), (3, 10, 1), (5, 2, 7), (5, 6, 3), (7, 2, 5), (7, 6, 1)$. Since student R reasons perfectly, he can establish that only these triples listed above are possible. In order to make his statement, r must be one of the uniquely mentioned values, 9 or 11. The triplet $(1, 2, 11)$ does not work because R would have known from P's statement that $p = 1$ and $q = 2$. Hence the only possible triplet is $(3, 2, 9)$ and the product is $3 \times 2 \times 9 = 54$