My Favorite Problems, 27

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This is column twentyseven of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

27.1 How many of the first Fibonacci numbers $F_1, F_2, \ldots, F_{1000}$ are multiples of 9?

27.2 How many 10 element subsets of $\{1, 2, 3, 4, \ldots, 100\}$ satisfy the property that no three of its members are the lengths of the sides of a non-degenerate triangle?

27.3 Suppose all six faces of an $a \times b \times c,\ a \leq b \leq c$ block of unit cubes are painted and it turns out that exactly the same number of unit cubes have some paint as those that have no paint. Find all triplets (a, b, c) for which this occurs. Prove that there are no other solutions. State and solve the planar analog of this problem.

Problems from My Favorite Problems, 26, with solutions.

26.1 Determine the unique pair of real numbers (x, y) that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

Solution: Solution by Robert Kotredes: The polynomial $4x^2 + 6x + 4$ can be written as $4(x + \frac{3}{4})^2 + \frac{7}{4}$ and therefore has a range of $[\frac{7}{4}, \infty)$. The polynomial $4y^2 - 12y + 25$ can be written $4(x - \frac{3}{2})^2 + 16$ and therefore has a range of $[16, \infty)$. Because $\frac{7}{4} \cdot 16 = 28$, the only possible values for each polynomial are their minimums, which occur at $x = -3/4$ and $y = 3/2$, respectively. So the unique pair of real numbers (x, y) is $(-\frac{3}{4}, \frac{3}{2})$.

26.2 Use the digits of the set $\{1, 3, 4, 6\}$ and some of the arithmetic operations $+, \times, -, \div$ to create the number 24.

Solution: One way to do this is $\frac{6}{1-\frac{3}{4}}$. I think this is unique.

26.3 John and Betty are the first to arrive at the game. They take their seats k and $k + 1$, and Betty notices that the seat numbers to her right, $1, 2, 3, \ldots, k - 1$ have a sum that is exactly one-seventh of the sum of the seat numbers to John's left. What assumptions do you need to assert that the solution is unique? What seat numbers do they occupy?

Solution: Seats 5 and 6. The sum of the seat numbers $1 + 2 + \cdots k - 1$ is $k(k - 1)/2$ and the sum of the seat numbers to John's left is $(n + k + 2)(n - k - 1)/2$ assuming there are n seats on the row. The simplified equation is

$$7k(k - 1) = n^2 + n - k^2 - 3k - 2.$$

Thus, by the quadratic formula,

$$\frac{-1 \pm \sqrt{32k^2 - 16k + 9}}{2}$$

which, by inspection is a perfect square 729 when $k = 5$. Thus $n = 13$ and Betty and John occupy seats 5 and 6. Indeed, $1 + 2 + 3 + 4 = \frac{1}{7}(7 + 8 + 9 + 10 + 11 + 12 + 13)$. Most rows of seats in stadiums have few than 50 seats, but $k = 28$ and $n = 78$ also work.