My Favorite Problems, 24

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This is column twentyfour of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

24.1 After each touchdown (and extra point), and also after each field goal, Sir Purr does a pushup for each Panther point on the scoreboard. For example, if the Panthers scored $3, 7$,and 3 points in that order, Sir Purr would do a total of $3 + 10 + 13 = 26$ pushups. Against the Steelers, Sir Purr had to do exactly 100 pushups. How many points could the Panthers have scored in the game. Assume they scored points in groups of 3 and 7 only.

24.2 How many ways can 1000 be expressed as a sum of powers of 2 if at most 2 of each power is allowed? For example, $8 = 4 + 4 = 4 + 2 + 2 = 4 + 2 + 1 + 1$, so there are four ways to write 8 as such a sum.

24.3 Use all ten digits exactly once to build three prime numbers with the least possible sum.

Problems from My Favorite Problems, 23, with solutions.

23.1 Find six different decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1$, but the sum is as large as possible. In this problem, a decimal digit is one of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ or 9.

Solution: One is $\frac{3}{7} + \frac{1}{8} + \frac{4}{9} = \frac{503}{504}$. This is the largest possible fraction because $504 = 7 \cdot 8 \cdot 9$ is the largest possible denominator. The number is roughly 0.9861.

23.2 Suppose a '+' sign or a '−' sign may be inserted in each of the eight positions of

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9.$$

For example, $123 - 4 - 5 - 6 + 7 + 8 - 9 = 114$. Can the number 100 be achieved?

Solution: Yes, $123 - 45 - 67 + 89 = 100$. Is this the only way? Here's another

$$1 + 2 + 3 - 4 + 5 + 6 + 78 + 9.$$

23.3 A rectangle with integral area has a perimeter of 13. How many different shapes are possible? Prove your answer.

Solution: Let x and y be the dimensions. Then xy must be an integer and $x + y = 6.5$. The graph of $A = x(6.5 - x)$ is a parabola that opens downwards and has a maximum value at $x = 3.25$. The vertex is $(3.25, 10.5625)$, so there are 10 different areas possible, hence 10 different shapes.