

My Favorite Problems, 23
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This is column twentythree of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at hbreiter@uncc.edu. In general, we'll list the problems in one issue and their solutions in the next issue.

23.1 Find six different decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1$, but the sum is as large as possible. In this problem, a decimal digit is one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9.

23.2 Suppose a '+' sign or a '-' sign may be inserted in each of the eight positions of

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9.$$

For example, $123 - 4 - 5 - 6 + 7 + 8 - 9 = 114$. Can the number 100 be achieved?

23.3 A rectangle with integral area has a perimeter of 13. How many different shapes are possible? Prove your answer.

Problems from My Favorite Problems, 22, with solutions.

- 22.1 A 10×10 square is decomposed into exactly 75 squares of various (integer) sizes. How many 3×3 squares are in this decomposition?

Solution: The largest square cannot be as large as 5×5 because in this case, the total number of squares would be less than 75. Let x, y, z , and w denote the number of squares of area 1, 4, 9, and 16 respectively. Then $x + 4y + 9z + 16w = 100$ and $x + y + z + w = 75$. Subtracting the latter from the former yields $3y + 8z + 15w = 25$. Since there are no integer solutions to $3y + 8z = 10$, we may conclude that $w = 0$. There is only one solution to $3y + 8z = 25$, namely, $y = 3$ and $x = 2$.

- 22.2 What is the fewest cuts needed to separate a wooden $3 \times 3 \times 3$ cube into 27 unit cubes if you're allowed to move blocks of cubes about before cutting? What if the big cube is $4 \times 4 \times 4$?

Solution: The fewest cuts needed for the $3 \times 3 \times 3$ is 6. To see this, note that the interior cube must be separated from each of its six neighbors by a cut. Oddly enough the minimum number of cuts required for the $4 \times 4 \times 4$ is also 6. Clearly, at least six are needed. But its not too hard to see that two cuts in each direction (with a move of blocks in between) is enough.

- 22.3 Let N be the huge number

$$N = 123456789101112 \dots 999$$

obtained by writing down, in order, the representation of the first 999 positive integers.

- (a) How many digits does N have?

Solution: The 9 single digit numbers require 9 digits; the 90 two digit numbers require 180 digits; the 900 three digit numbers require 2700 digits. So it take $9+180+2700 = 2889$ digits to write all of N .

- (b) How many times does the digit 6 appear in N ?

Solution: 300. In fact each nonzero digit occurs every tenth time as the units digit of successive integers. Also, within each 100 consecutive integers, each non-zero digit occurs ten times. Similarly, each non-zero digit occurs 100 times as the hundreds digit in N . Therefore the digit 6 appears $100 + 100 + 100 = 300$ times in N .

- (c) What is the product of the 2009th digit and the 2010th digit of N ?

Solution: Since there are 189 digits needed to write the single-digit and double-digit numbers, we need to write $2009 - 189 = 1820$ more digits. Dividing by 3, we see that we are writing the 606th three-digit number, which is the number 605. Since $606 \cdot 3 = 1818$, the 2008th digit is 6 and the 2009th digit is 0, so the product is zero.