My Favorite Problems, 22

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This is column twentytwo of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

22.1 A 10×10 square is decomposed into exactly 75 squares of various (integer) sizes. How many 3×3 squares are in this decomposition?

22.2 What is the fewest cuts needed to separate a wooden $3 \times 3 \times 3$ cube into 27 unit cubes if you're allowed to move blocks of cubes about before cutting? What if the big cube is $4 \times 4 \times 4$?

22.3 Let N be the huge number

$$N = 123456789101112\ldots999$$

obtained by writing down, in order, the representation of the first 999 positive integers.

(a) How many digits does N have?

(b) How many times does the digit 6 appear in N?

(c) What is the product of the 2009^{th} digit and the 2010^{th} digit of N?

Problems from My Favorite Problems, 21, with solutions.

21.1 The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Use this information to find the exact location of all the mines. A square can have more than one mine.

	1		1		2		2		1
1		2		3		2		3	
	4		2		4		2		2
2		4		3		2		3	
	4		2		2		1		2
2		4		1		2		2	
	3		3		3		4		1
1		2		2		4		2	

Solution: The solution is unique. First note that no square contains as many as three mines. Give reasons for why that is true. Then start with a conjecture that there is a mine in the upper left corner. Call that position $(0,0)$. Then the diagonal of squares $(0,-2),(-1,-1),(2,0)$ must all be empty. Write 1 in the upper left corner and zeros in the three squares $(0,-2),(-1,-1),(2,0)$. Then build outwards from that to get a contradiction. Now start with a zero in position $(0,0)$, and again build outward. You'll be forced to put two mines at $(1,-3)$. Then you're in business, and the rest of grid is forced.

	1		1	$\bullet$	2		2	$\bullet$	1
1	$\bullet$	2		3	$\bullet$	2	$\bullet$	3	
	4	$\bullet$	2	$\bullet$	4		2	$\bullet$	2
2	$\bullet\bullet$	4		3	$\bullet\bullet$	2		3	$\bullet$
	4	$\bullet$	2		2		1	$\bullet$	2
2	$\bullet$	4	$\bullet$	1		2		2	
$\bullet$	3	$\bullet$	3		3	$\bullet\bullet$	4	$\bullet$	1
1		2	$\bullet$	2	$\bullet$	4	$\bullet$	2	

21.2 How many numbers can be achieved by putting + signs into 1 2 3 4 5 6 7? For example, one such number is $12 + 3 + 4 + 5 + 6 + 7 = 37$.

Solution: The answer is 56. First note that there are 64 ways to insert plus sign in the six positions. We can partition this set of numbers into subsets based on the number of plus signs. In fact, the sixth row 1 6 15 20 15 6 1 of Pascal's triangle tells us just how to do this. There is one string with no plus signs, 6 with one, 15 with 2 , 20 with 3 plus signs, etc. A little inspection shows that duplication only occurs when there are 3, 4 or 5 plus signs. With six plus signs, we have $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. The removal of a plus sign leave a numeral with the same sum of digits. Thus, the removal of a plus sign increases the value of the expression by a multiple of 9. We can obtain $\{28 + 9k \mid k = 0, \ldots, 6$ by removing none or one of the plus signs. Removing two non-adjacent plus signs results in duplication. Making a list of achievable numbers shows that the numbers $28, 37, 46$, and 55 are uniquely achievable. Continuing $64 = 12 + 34 + 5 + 6 + 7 = 1 + 2 + 3 + 45 + 6 + 7$ and $73 = 1+2+3+4+56+7 = 12+3+45+6+7$. The number 82 can be achieved in 3 ways, 91 in 2, 100 in 2, 109 in 2, 118 in 2. The smallest number that includes a 3-digit summand is $123 + 4 + 5 + 6 + 7 = 145$ which is bigger than the largest number without such a summand $1+23+45+67 = 136$, so we have found all the duplicates. obtainable Since there are $2^6 = 64$ ways to insert pluses, there are $64 - 8 = 56$ different numbers attainable.

21.3 The World Series is a best-of-seven series of baseball games used to pick the Major League Baseball champion. If team A wins with probability p and team B wins with probability $q = 1 - p$, find the probability that the series lasts exactly $4, 5, 6$ and 7 games.

Solution: Note that $1 = 1^4 = (p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$, and each term of this polynomial represents a sequence of four games. Of course p^4 and q^4 represent the probability that A and B respectively win in four games. Now the probability that the series goes more than 4 games is $1-p^4-q^4 = 1\cdot 4p^3q+6p^2q^2+4pq^3 = (p+q)(4p^3q+6p^2q^2+4pq^3) = 4p^4q + 10p^3q^2 + 10p^2q^3 + 4pq^4$. The first and last terms represent five game series and the two middle terms the rest. Thus the series ends after five games with probability $4p^4q + 4pq^4$ and goes to at least six games with probability $1 \cdot (10p^3q^2 + 10p^2q^3) = (p+q)\cdot(10p^3q^2 + 10p^2q^3) = 10p^4q^2 + 20p^3q^3 + 10p^2q^4$, the middle term of which is the probability that the series requires seven games.