My Favorite Problems, 21

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This is column twentyone of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

21.1 The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Use this information to find the exact location of all the mines. A square can have more than one mine.

	1		1		2		2		1
1		2		3		2		3	
	4		2		4		2		2
2		4		3		2		3	
	4		2		2		1		2
2		4		1		2		2	
	3		3		3		4		1
1		2		2		4		2	

21.2 How many numbers can be achieved by putting + signs into 1 2 3 4 5 6 7? For example, one such number is $12 + 3 + 4 + 5 + 6 + 7 = 37$.

21.3 The World Series is a best-of-seven series of baseball games used to pick the Major League Baseball champion. If team A wins with probability p and team B wins with probability $q = 1 - p$, find the probability that the series lasts exactly $4, 5, 6$ and 7 games.

Problems from My Favorite Problems, 20, with solutions.

20.1 A positive integer c bigger than 1 can be *split* into two positive integer summands a and b. The *value* of the split is ab. For example the value of the split of 7 into 2 and 5 is $2 \cdot 5 = 10$. The number 20 is written on a board. The splitting process is repeated 19 times until the only unsplit numbers are 1's. What is the greatest possible sum of the values of the 19 splits?

Solution: (I heard this problem from Joshua Zucker) The answer is 190. In fact the sum of the values of the 19 splits is 190 independent of the way the splitting process proceeds. For any number n, the sum of the values of the $n-1$ splits is the triangular number $t_{n-1} = (n-1)n/2$. To see this, suppose that a group of 20 people are together in a room. A non-empty subset leaves the rest in the room and, as they leave, each person leaving shakes hands with each person staying. This subset formation followed by shaking of hands continues until each person occupies a room by himself. At this point, each person has shaken hands with each other person.

20.2 (Purple Comet 2005) Let k be the product of every third positive integer from 2 to 2006, that is, $k = 2 \cdot 5 \cdot 8 \cdots \cdot 2006$. Find the number of zeros there are at the right end of the decimal representation of k.

Solution: Of course, we have plenty of 2's in the prime factorization to match all the 5's. The exponent on the 5 in the prime factorization of the number is 168.

20.3 You're given 27 unpainted cubes. Can you paint the faces with three colors, red, white, and blue, so that when you're done, you can assemble an all red $3 \times 3 \times 3$ cube, an all white $3 \times 3 \times 3$ cube and an all blue $3 \times 3 \times 3$ cube?

Solution: Yes, you can do this in essentially just one way.

Since the total number of faces we can paint is $27 \cdot 6 = 162$, and since the all-red cube requires $6 \cdot 9 = 54$ red faces, we must be perfectly efficient in the following sense. Each face we paint red must appear on the outside of the red cube. This implies that there must be exactly one unit cube that has no red faces, and similarly exactly one that has no white, and one that has no blue. These cubes must have exactly three faces of the other two colors. They all look like $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{x}} & & \\ \hline \multicolumn{1}{|c|}{\text{x}} & \multicolumn{1}{c|}{\text{x}} & \multicolumn{1}{c|}{\text{y}} & \multicolumn{1}{c|}{\text{y}} \\ \hline & \multicolumn{1}{|c|}{\text{y}} & & \\ \cline{2-2}\end{array}$, where x is one color and y is another. The two cubes of this type with red faces account for two of the eight corners. The six other red corner cubes must have faces with the two other colors, so two faces must be one color and one face the third color. There must be three cubes with $3R, 2W, 1B$ and three more with $3R, 1W, 2B$.

That is three cubes of each pattern $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{R}} & \multicolumn{1}{c|}{\text{R}} & \multicolumn{1}{c|}{\text{W}} & \multicolumn{1}{c|}{\text{W}} \\ \hline & \multicolumn{1}{|c|}{\text{B}} & & \\ \cline{2-2}\end{array}$ and $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{R}} & \multicolumn{1}{c|}{\text{R}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{W}} \\ \hline & \multicolumn{1}{|c|}{\text{B}} & & \\ \cline{2-2}\end{array}$. So the red corners are done. There must be 12 red edge cubes, that is 12 cubes with just two red, adjacent faces. There are two types of these $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{R}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{W}} & \multicolumn{1}{c|}{\text{W}} \\ \hline & \multicolumn{1}{|c|}{\text{W}} & & \\ \cline{2-2}\end{array}$ and $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{R}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{W}} \\ \hline & \multicolumn{1}{|c|}{\text{B}} & & \\ \cline{2-2}\end{array}$. There are three of each of these. The other six red edges are of the type $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{R}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{W}} \\ \hline & \multicolumn{1}{|c|}{\text{W}} & & \\ \cline{2-2}\end{array}$. Finally we need six cubes that can serve as face cubes in the $3 \times 3 \times 3$ red cube. These must have exactly one red face. There are two types of these, $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{B}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{W}} & \multicolumn{1}{c|}{\text{W}} \\ \hline & \multicolumn{1}{|c|}{\text{W}} & & \\ \cline{2-2}\end{array}$ and $\begin{array}{cccc}\cline{2-2} & \multicolumn{1}{|c|}{\text{R}} & & \\ \hline \multicolumn{1}{|c|}{\text{W}} & \multicolumn{1}{c|}{\text{W}} & \multicolumn{1}{c|}{\text{B}} & \multicolumn{1}{c|}{\text{B}} \\ \hline & \multicolumn{1}{|c|}{\text{B}} & & \\ \cline{2-2}\end{array}$. You can see from the symmetry that all three monochromatic cubes can be built.