My Favorite Problems, 20

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This is column twenty of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

20.1 A positive integer c bigger than 1 can be *split* into two positive integer summands a and b. The *value* of the split is ab. For example the value of the split of 7 into 2 and 5 is $2 \cdot 5 = 10$. The number 20 is written on a board. The splitting process is repeated 19 times until the only unsplit numbers are 1's. What is the greatest possible sum of the values of the 19 splits?

20.2 (Purple Comet 2005) Let k be the product of every third positive integer from 2 to 2006, that is, $k = 2 \cdot 5 \cdot 8 \cdots 2006$. Find the number of zeros there are at the right end of the decimal representation of k.

20.3 You're given 27 unpainted cubes. Can you paint the faces with three colors, red, white, and blue, so that when you're done, you can assemble an all red $3 \times 3 \times 3$ cube, an all white $3 \times 3 \times 3$ cube and an all blue $3 \times 3 \times 3$ cube?

Problems from My Favorite Problems, 19, with solutions.

19.1 (2008 Tournament of the Towns) Does there exist a permutation a_i of the positive integers so that the sum of each segment $\sum_{i=m}^{i=n} a_i$ is a composite number for all $m < n$.

Solution: Start with $a_1 = 1$ and $a_2 = 3$. Now for each sequence $a_1, a_2, \ldots, a_{2n-1}, a_{2n}$ construct the next two members of the sequence as follows: $a_{2n+2} = mex\{a_1, a_2, \ldots, a_{2n}\}$, that is the smallest positive integer not in the set $\{a_1, a_2, \ldots, a_{2n}\}$. Then choose a_{2n+1} as the smallest positive integer not in the set $\{a_1, a_2, \ldots, a_{2n}, a_{2n+2}\}$ for which all $4n+1$ numbers $\sum_{i=m}^{i=t} a_i$ are composite, where $t = 2n+1$ or $t = 2n+2$. In other words, a_{2n+1} is the smallest available positive integer whose inclusion does not create any prime *summed segments* among the first $2n+2$ members of the sequence. By the principle of mathematical induction, the sequence is a permutation of the positive integers, no segment of which has a prime number sum.

19.2 Let f be a cubic polynomial with quadratic term zero; that is, $f(x) = a + bx + cx^3$, with $c \neq 0$. Let a and b be positive real numbers. Let $L = mx + b$ denote the line through the points $(-a, f(-a))$ and $(-b, f(-b))$. The L intersects the graph of f at the point $(c, f(c))$. prove that $a + b = c$.

Solution: To prove that $a + b = c$, note that $g(x) = f(x) - mx - b$ is a cubic polynomial with quadratic term zero and two roots $x = -a$ and $x = -b$. Since the sum of the zeros of g is 0, it follows that the $x = a + b$ is also a zero of g.

19.3 (Purple Comet 2005) A tailor met a tortoise sitting under a tree. When the tortoise was the tailor's age, the tailor was only a quarter of his current age. When the tree was the tortoises age, the tortoise was only a seventh of its current age. If the sum of their ages is now 264, how old is the tortoise?

Solution: The tortoise is 77 years old. Let a, o, t represent the ages of the tailor, the tortoise, and the tree respectively. Then $4(2a - o) = a, 7(2o - t) = o$, and $a + o + t = 264$. Solving simultaneously, we get $o = 77, a = 44$, and $t = 143$.