My Favorite Problems, 19

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This is the nineteenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

19.1 (2008 Tournament of the Towns) Does there exist a permutation a_i of the positive integers so that the sum of each segment $\sum_{i=m}^{i=n} a_i$ is a composite number for all $m < n$.

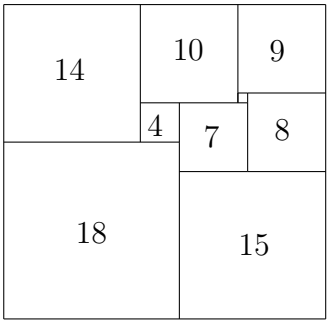
19.2 Let f be a cubic polynomial with quadratic term zero; that is, $f(x) = a + bx + cx^3$, with $c \neq 0$. Let a and b be positive real numbers. Let $L = mx + b$ denote the line through the points $(-a, f(-a))$ and $(-b, f(-b)$. The L intersects the graph of f at the point $(c, f(c))$. prove that $a + b = c$.

19.3 (Purple Comet 2005) A tailor met a tortoise sitting under a tree. When the tortoise was the tailor's age, the tailor was only a quarter of his current age. When the tree was the tortoises age, the tortoise was only a seventh of its current age. If the sum of their ages is now 264, how old is the tortoise?

Problems from My Favorite Problems, 18, with solutions.

18.1 Construct a rectangle by putting together nine squares with sides equal to 1, 4, 7, 8, 9, 10, 14, 15 and 18.

Solution: Factor the sum of the squares to get $1^2 + 4^2 + 7^2 + \cdots + 18^2 = 1056 = 2^5 \cdot 3 \cdot 11$. Notice that the only pair of dimensions that will accommodate the 18×18 square together with both the 14×14 and the 15×15 squares is 32×33. The four corners are unique. The only way to make room for the three largest squares is to put them in corners with the 14×14 square and the 15×15 square next to the 18×18 square. See the figure below. The only two squares that could fill the 3×15 gap left above the 15×15 square are the 7×7 and the 8×8 squares. Then the 1×1 must go in the tiny hole left. Finally the 10×10 and the 9×9 squares can be placed.



18.2 Suppose $(S, 0, +)$ is a finite Abelian group on the set S, and $\cdot$ is a commutative binary operator on S. Also, suppose $(S, 0, +)$ distributes over $(S, \cdot)$. That is, $\forall a, b, x \in S,\ x + (a \cdot b) = (x + a) \cdot (x + b)$.

(a) Show that $|S|$ is odd.

(b) Also, given $(S, 0, +)$, find all binary operators $\cdot$ that satisfy these conditions.

Solution: First, suppose $\forall x \in S, 0 \cdot x$ is known.

Then $\forall a, b \in S, a \cdot b = (0 + a) \cdot (b - a + a) = [0 \cdot (b - a)] + a$.

Therefore, $(S, \cdot)$ is completely defined from the values $0 \cdot x, x \in S$.

Now since $\forall a, b \in S, a \cdot b = b \cdot a$ we know that $\forall a, b \in S, a \cdot b = [0 \cdot (b - a)] + a = [0 \cdot (a - b)] + b = b \cdot a$.

Letting $b - a = x$, we see that $0 \cdot (-x) = [0 \cdot x] - x$.

Therefore, $(S, \cdot)$ is commutative if and only if $\forall x \in S, 0 \cdot (-x) = (0 \cdot x) - x$. This condition determines all commutative $(S \cdot)$ such that $(S, 0, +)$ distributes over $(S \cdot)$.

Now suppose $|S|$ is even. Then since $(S, 0, +)$ is an Abelian group $\exists x \in S$ such that $x \neq 0$ and $x + x = 2x = 0$. This means $x \neq 0$ and $-x = x$. But then $0 \cdot (-x) = [0 \cdot (x)] - x$ is impossible since $0 \cdot (-x) = 0 \cdot (x)$.

Note. If $|S|$ is odd, then $\forall x \in S \setminus \{0\}, x \neq -x$.

18.3 Consider the $a \times b \times c$ rectangular box built from abc unit cubes, where a, b, and c are positive integers. How many paths of length $a + b + c$ are there from a fixed corner of the box to the corner farthest away along edges of the unit cubes that stay on the surface of the box?

Solution: This solution is due to Kathleen E. Lewis, SUNY Oswego, Oswego, NY. Suppose the box is put into three-dimensional coordinate space with one vertex at $(0, 0, 0)$ and the opposite vertex at (a, b, c). To specify a path along the edges from the origin to the opposite vertex, we need a sequence of length $a + b + c$ containing a copies of x, b copies of y and c copies of z. All such sequences will produce paths along the edges of the unit cubes, but not all will stay on the surface of the box. In order for a point (r, s, t) to be on the surface, at least one of the three coordinates must be either zero or the maximum value for that coordinate. Therefore, a path will only stay on the surface if at least one variable reaches its maximum value while one of the other variables is still zero. This means that the corresponding sequence has all occurrences of one of the variables before any of the occurrences of the third variable. For instance, if the first coordinate reaches a while the third coordinate is still zero, the sequence would have all x's appearing before the first z. Such a sequence could be seen as the merger of a sequence of length $a + c$ containing x's followed by z's and a sequence of b copies of y. There are $\binom{a+b+c}{b}$ such sequences, since the location of the y's determine the entire sequence. Similarly, there are $\binom{a+b+c}{c}$ sequences with all the x's appearing before any of the y's. Looking at all the possible combinations, we would seem to get

$$2\left[\binom{a+b+c}{a} + \binom{a+b+c}{b} + \binom{a+b+c}{c}\right]$$

sequences. But some sequences have been counted more than once, so this number is too large. For instance, a sequence in which all of the x's appear as a block at the beginning, followed by a mixture of y's and z's could have resulted from an xz sequence merged with a y sequence or an xy sequence merged with a z sequence. There are $\binom{b+c}{b}$ such sequences, since only the y's and z's need to be arranged. Similarly, there are equally many sequences in which all the occurrences of x are at the end, with the y's and z's mixed together. In all, there are

$$2\left[\binom{a+b}{a} + \binom{b+c}{b} + \binom{c+a}{c}\right]$$

such sequences which have been double-counted. Thus, a more accurate count would be

$$2\left[\binom{a+b+c}{a} + \binom{a+b+c}{b} + \binom{a+b+c}{c} - \binom{a+b}{a} - \binom{b+c}{b} - \binom{c+a}{a}\right],$$

since this only counts those sequences once. But what about the sequences in which each of the three letters is grouped in a block? They would have been counted three times in the original count, but then subtracted twice, so they end up being counted exactly once, as they should be. So, this last formula gives the number of paths that stay on the surface of the cube.