My Favorite Problems, 15

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This is the fifteenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

15.1 (This beautiful problem is due to Andy Niedermaier, a graduate student at University of California San Diego) Consider a 10×10 grid of lights, each either on or off, which we denote using matrix notation $a_{i,j}$, where, for each $i = 1, 2, \ldots, 10$ and $j = 1, 2, \ldots, 10$, the entry in row i and column j is $a_{i,j}$ and its value is 0 or 1. We are allowed two types of *moves*. For each $1 \leq u \leq 8$ and $1 \leq v \leq 8$, we can change the status of all the lights $a_{i,j}$ for which both $u \leq i \leq u + 2$ and $v \leq j \leq v + 2$. This is called a *small block* move. The other type move is, for each $1 \leq u \leq 6$ and $1 \leq v \leq 6$, we can change the status of all the lights $a_{i,j}$ for which both $u \leq i \leq u+4$ and $v \leq j \leq v+4$. This is called a *large block* move. So essentially, we can change the status of all nine lights in each 3×3 subarray and of all the lights in each 5×5 subarray. Is it possible, beginning with the all on configuration, to achieve any configuration of lights?

15.2 Solve the equation $\sqrt{5 - x} = x^2 - 5$ over the reals.

Problems from My Favorite Problems, 14, with solutions.

14.1 Find six different nonzero decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1$ and the sum is as large as possible.

Solution: The unique solution is $\frac{3}{7} + \frac{1}{8} + \frac{4}{9} = \frac{503}{504}$. Note that the sum $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ is a fraction whose denominator is at most $7 \cdot 8 \cdot 9 = 504$, so the largest possible fraction less than 1 is $503/504$. The problem is to find a, c and e such that $\frac{a}{7} + \frac{c}{8} + \frac{e}{9} = 503/504$, which after some algebra leads to $72a + 63c + 56e = 503$. Note that the only non-multiple of 9 on the left side is $56e$ which is congruent to $2e$. The right side 503 is 8 larger than a multiple of 9, ie is congruent to 8 (mod 9). so e must be 4. That is $72a + 63c = 503 - 56 \cdot 4 = 279$. Dividing both sides by 9 yields $8a + 7c = 31$, which we can solve by inspection, $a = 3$ and $c = 1$. Thus $\frac{3}{8} + \frac{1}{7} + \frac{4}{9} = 503/504$ is the largest possible value less than 1 that the expression can have.

14.2 What is the volume of the polyhedron Q defined by $|z-3|+|x-y|+|x+y|+|x|+|y| \leq 6$?

Solution: Let $z=3$. Let P denote the polygonal solution to

$$* \qquad |x-y|+|x+y|+|x|+|y|=6.$$

Note that P is symmetric with respect to each of $x=0$, $y=0$ and $(0,0)$. If $0 \leq y \leq x$, the line segment from $(2,0)$ to $(3/2,3/2)$ satisfies $*$. It follows that P is a convex octagon with vertices $(\pm 2,0),(0,\pm 2),(\pm 3/2,\pm 3/2)$ and area $16-4=12$.

Now Q is a pair of attached cones over P, attached at a copy of P three units above the x-y plane. This 10 vertex polyhedron has extreme points $(\pm 2,0,3),(0,\pm 2,3),(\pm 3/2,\pm 3/2,3),(0,0,9)$ and $(0,0,-3)$. The volume V of Q is given by $V=2\cdot 1/3\cdot 12\cdot 6=48$.

14.3 Let C denote the 16-element set $\{(a_1,a_2,a_3,a_4) \mid a_i \in \{0,1\}, i=1,2,3,4\}$ in Euclidean space E_4. Let T denote the set of all triangles all of whose vertices belong to C. How many members of T are acute? How many members of T are right triangles? How many members of T are obtuse?

Solution: Every three element subset of C determines a triangle because no three members of C are colinear. There are $\binom{16}{3}=560$ three-element subsets. For each triangle ABC, we define the *shape* abc, $a \leq b \leq c$, where a,b and c represent the number of coordinates in which two points differ. For example, the shape of the triangle with vertices $(0,0,0,0),(1,0,0,0)$ and $(0,1,1,1)$ is 134. There are exactly six different shapes: 112, 123, 134, 224, 222, and 233. None are obtuse, by the converse of the Pythagorean Theorem. The first four shapes are right triangles and the last two acute. Now each of the 16 vertices belongs to exactly 105 triangles since there are $3\cdot 560=1680$ vertices among the 560 triangles and each vertex appears the same number of times as any other. Therefore, we can simply count the triangles of each shape that include the origin $O=(0,0,0,0)$. In what follows, it is useful to know the number of members of C that differ in 1, 2, 3 and 4 places from O. This is easily seen to be $4,6,4,1$ respectively. So, to count the number of triangles of shape 112 that include O, we note that there are two types, one in which the right angle is at O and the other where it is not. There are $\binom{4}{2}=6$ of the former and $4\cdot 3=12$ of the latter. Continuing in this way, there are 36 of the shape 123, 12 of shape 134 and 9 of shape 224. There are 12 of shape 222 and 18 of shape 233 for a total of 105. Since $12+18=30$ of the 105 triangles with a vertex O are acute, 160 of the 560 triangles are acute and the other 400 are right triangles.