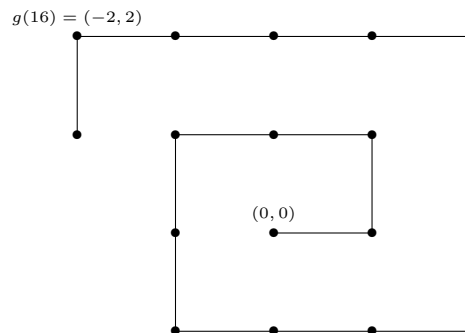


My Favorite Problems, 13
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This is the thirteenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at hbreiter@email.uncc.edu. In general, we'll list the problems in one issue and their solutions in the next issue.

- 13.1 The three points $(4, 14, 8, 14)$, $(6, 6, 10, 8)$ and $(2, 4, 6, 8)$ are vertices of a three dimensional cube in 4-space. Find the center of the cube.
- 13.2 Let $n \geq 1$ be fixed. Suppose n points are placed at random on a circle. Let $P(n)$ denote the probability that all n points lie on the same side of some diameter? In particular, find $P(2)$ and $P(3)$.
- 13.3 The bug is back, again crawling around the plane at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1, 0)$, turns 90° left and crawls another unit to $(1, 1)$, turns 90° left again, and crawls two units. He continues to make 90° left turns as shown in the figure. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.) Let $g(t)$ denote the position in the plane after t minutes, where t is an integer. Thus, for example, $g(0) = (0, 0)$, $g(6) = (-1, -1)$, and $g(16) = (-2, 2)$. Does there exist an integer t such that $g(t)$ and $g(t + 23)$ are exactly 17 units apart? If so, find the smallest such t .



Problems from My Favorite Problems, 12, with solutions.

- 12.1 Peter tosses 25 fair coins and John tosses 20 fair coins. What is the probability that they toss the same number of heads?

Solution: This little jewel came to me from Alex Gordon, University of North Carolina Charlotte. It is well-known that the number k of successes in n independent tries of a repeated experiment with probability p of success (and therefore $q = 1 - p$ of failure) is given by the binomial theorem:

$$P(x = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

In the case of a fair coin, $p = q = \frac{1}{2}$, so the probability can be written $P(x = k) = \binom{n}{k} / 2^n$. Now Peter and John could both toss no heads, or both 1 head, or both 2, ..., or both 20 heads, so the answer can be expressed as a sum of products,

$$P(x = y) = \sum_{i=0}^{20} P(x = y = i) = \sum_{i=0}^{20} \left(\binom{25}{i} / 2^{25} \right) \left(\binom{20}{i} / 2^{20} \right).$$

But $P(x = k) = P(x = n - k)$ since heads and tails happen with the same probability. Changing John's probability in that way gives

$$P(x = y) = \sum_{i=0}^{20} P(x = y = i) = \sum_{i=0}^{20} \left(\binom{25}{i} / 2^{25} \right) \left(\binom{20}{20-i} / 2^{20} \right) = P(x = 20) = \binom{45}{20} / 2^{45}.$$

This number is approximately 0.090093.

- 12.2 Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Is it possible that the number of unit cubes with some red paint is exactly $M = 2006$ larger than the number of cubes with some black paint? Find the smallest number $M \geq 2006$ for which there is such a cube and find a way to paint the faces so that the number of unit cubes with some red paint is exactly M larger than the number of cubes with some black paint.

Solution: The answer is $M = 2046$. We paint black an opposite pair of faces of a $33 \times 33 \times 33$ cube. Note that the number r of red faces must be four, five, or six. There are two cases with four red faces: the black faces are adjacent or not. The four equations we get are $r = 6 : n^2 - 12n + 8 = M$; $r = 5 : 4(n - 1)^2 = M$; $r = 4 : 2(n - 1)^2 = M$ and $2n(n - 2) = M$, none of which have integer solutions for $2006 \leq M \leq 2045$.

12.3 (London Sunday Times, July 2006) At ABC University, the mascot does as many pushups after each ABCU score as the team has accumulated. The team always make extra points after touchdowns, so it scores only in increments of 3 and 7. For each sequence a_1, a_2, \dots, a_n where each a_k is 3 or 7, let $P(a_1, a_2, \dots, a_n)$ denote the total number of pushups the mascot does for the scoring sequence a_1, a_2, \dots, a_n . For example $P(3, 7, 3) = 3 + (3 + 7) + (3 + 7 + 3) = 26$. Call a positive integer k *accessible* if there is a scoring sequence a_1, a_2, \dots, a_n such that $P(a_1, a_2, \dots, a_n) = k$. Is there a number K such that for all $t \geq K$, t is accessible. If not, prove it and if so, find K .

Solution: The number of pushups the mascot must do is $7x + 3(T_n - x)$ where $T_n = 1 + 2 + \dots + n$, when the team scores n times during the game. Now the remainders when $T_n = n(n + 1)/2$ is divided by 4 are 1, 3, 2, 2, 3, 1, 0, 0, and this sequence repeats. In other words, $T_1 \equiv 1 \pmod{4}$, etc. Let $K_n = \{7x + 3(T_n - x) \mid x = 0, 1, \dots, n\}$. Note that K_n is the number of possible pushups the mascot must complete when his team scores n times. For example $K_1 = \{3, 7\}$, $K_2 = \{9, 13, 17, 21\}$. Its easy to see that the elements of each K_n all differ by a multiple of 4 from one another. Thus, we have four problems, one for each remainder r .

For $r = 1$, we compute $K_1, K_6, K_9, K_{14}, K_{17}$, etc. Since $K_1 = \{3, 7\}$ and $K_6 = \{63, 67, \dots, 147\}$ and $K_9 = \{135, 139, \dots, 315\}$ and $K_{14} = \{315, 319, \dots, \}$ the largest unacheivable number is 59.

For $r = 2$, we must compute K_3, K_4, K_{11}, K_{12} and K_{19} . The largest member of K_{12} is $7 \cdot 12 \cdot 13/2 = 546$ and the smallest member of K_{19} is $3 \cdot 19 \cdot 20/2 = 570$, so 566 is the largest unacheivable number of this sort.

For $r = 3$, we have $K_2 = \{9, 13, 17, 21\}$, $K_5 = \{45, 49, 53, \dots, 105\}$, $K_{10} = \{165, 169, \dots, 385\}$, and K_{13} has smallest member 273, we can see that 161 is the largest unacheivable number of pushups for which the remainder is 2.

Finally, for $r = 0$, a similar analysis shows that the largest unacheivable number of this sort is 116. Therefore, 566 is the largest unacheivable number.