

My Favorite Problems, 12
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This is the twelfth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at hbreiter@email.uncc.edu. In general, we'll list the problems in one issue and their solutions in the next issue.

- 12.1 Peter tosses 25 fair coins and John tosses 20 fair coins. What is the probability that they toss the same number of heads?
- 12.2 Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Is it possible that the number of unit cubes with some red paint is exactly $M = 2006$ larger than the number of cubes with some black paint? Find the smallest number $M \geq 2006$ for which there is such a cube and find a way to paint the faces so that the number of unit cubes with some red paint is exactly M larger than the number of cubes with some black paint.
- 12.3 (London Sunday Times) At ABC University, the mascot does as many pushups after each ABCU score as the team has accumulated. The team always make extra points after touchdowns, so it scores only in increments of 3 and 7. For each sequence a_1, a_2, \dots, a_n where each a_k is 3 or 7, let $P(a_1, a_2, \dots, a_n)$ denote the total number of pushups the mascot does for the scoring sequence a_1, a_2, \dots, a_n . For example $P(3, 7, 3) = 3 + (3 + 7) + (3 + 7 + 3) = 26$. Call a positive integer k *accessible* if there is a scoring sequence a_1, a_2, \dots, a_n such that $P(a_1, a_2, \dots, a_n) = k$. Is there a number K such that for all $t \geq K$, t is accessible. If not, prove it and if so, find K .

Problems from My Favorite Problems, 11, with solutions.

- 11.1 A father wishes to take his two sons to visit their grandmother who lives 6.5 miles away. His motorcycle has a top speed of 35 miles per hour. With one passenger it drops to 25 mph. He cannot carry more than one passenger. Each boy walks at 5 mph. Show that the father can get them all there in not more than half an hour.

Solution: The father can take one son far enough so that the son can walk the remaining distance in a total of 30 minutes. He must choose d so that d satisfies $d/25 + (6.5 - d)/5 = 0.5$, or $d = 5$. Now if $d > 5$, then that son will arrive before half an hour has elapsed, but the other one can't. So after $5/25 = 0.2$ hours, the father drops off the first son and turn around to get the second, who has walked $0.2 \cdot 5 = 1$ mile. Now the father and the second son are four miles apart and converging at the rate of $35 + 5$ miles per hour. It takes them $4/40 = 0.1$ hours to meet, by which time the second son has walked another half mile. At this point they have exactly 5 miles to granny's and they have 0.2 hours to make the trip, just enough time at 25 miles per hour.

- 11.2 (2005 Michigan Math Take Home Challenge) An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

Solution: The number of edges is $\frac{1}{2}(15 \cdot 4 + 7 \cdot 5 + 15 \cdot 3) = 70$, so by Euler's formula $e + 2 = f + v$, we have $70 + 2 = 37 + v$ and $v = 35$.

- 11.3 (2005 Michigan Math Take Home Challenge) Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

Solution: 2005. Let $G(n)$ denote the number of ways to write n as a sum of approximately equal positive integers. Trial and error produces $G(1) = 1, G(2) = 2, G(3) = 3$ and $G(4) = 4$. In fact, we can prove by mathematical induction that $G(n) = n$ for all positive integers. Suppose $n = a_1 + a_2 + \cdots + a_k$ where either all the a_i are the same or there are two different values and they differ by 1. Thus we have either $n = ka + m(a - 1), m > 0$ or $n = ka$. In the first case $n + 1 = (k + 1)a + (m - 1)(a - 1)$ if $m > 0$ and $n + 1 = a + 1 + (k - 1)a$. So we have a bijection between the representations of n and the representations of $n + 1$ that include at least one number other than 1. In addition $n + 1 = 1 + 1 + 1 + \cdots + 1$ represents a new representation. Thus $G(n + 1) = G(n) + 1$ for all integers n .

Alternatively, note that for each $n, 1 \leq n \leq 2005$ there is exactly one sum with n terms. To see this, divide 2005 by n . If $2005 = nq + r$, then $2005 = (n - r)q + r(q + 1)$. Of course q and $q + 1$ are approximately equal.