My Favorite Problems, 11

Harold B. Reiter
University of North Carolina Charlotte

This is the eleventh of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

11.1 (Clayton Dodge, 2001 Michigan Math Challenge) A father wishes to take his two sons to visit their grandmother who lives 6.5 miles away. His motorcycle has a top speed of 35 miles per hour. With one passenger it drops to 25 mph. He cannot carry more than one passenger. Each boy walks at 5 mph. Show that the father can get them all there in not more than half an hour.

11.2 (2005 Michigan Math Take Home Challenge) An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

11.3 (2005 Michigan Math Take Home Challenge) Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

Problems from My Favorite Problems, 10, with solutions.

10.1 (This is problem appears in The Art and Craft of Problem Solving (Problem 7.2.13), by Paul Zeitz.) A sequence of integers is defined by $a_0 = p$, where $p > 0$ is a prime number, $a_n + 1 = 2a_n + 1$, for $n = 0, 1, 2, \ldots$. Is there a value of p such that the sequence consists entirely of prime numbers?

Solution: First, find a closed form solution for a_n as follows. Add 1 to the recurrence relation to get $a_{n+1} + 1 = 2(a_n + 1)$. Hence $a_n + 1 = 2^n(a_0 + 1)$. Substituting $a_0 = p$, we obtain $a_n = 2^n p + (2^n - 1)$.

If $p = 2$, then $a_1 = 5$ is prime, and so, without loss of generality, we may assume that p is odd. Consider the congruence $a_n \equiv 2^n - 1 (\text{mod } p)$. Then, since 2 and p are relatively prime, by Fermat's Little Theorem, $2^{p-1} \equiv 1 (\text{mod } p)$. Hence $a_{p-1} \equiv 0 (\text{mod } p)$. Since $a_{p-1} > p$, it follows that a_{p-1} is composite.

Therefore, there is no value of p such that the above sequence consists entirely of prime numbers.

10.2 Let $S(n)$ denote the sum of the decimal digits of of the integer n. For example $S(64) = 10$. Find the smallest integer n such that

$$S(n) + S(S(n)) + S(S(S(n))) = 2007.$$

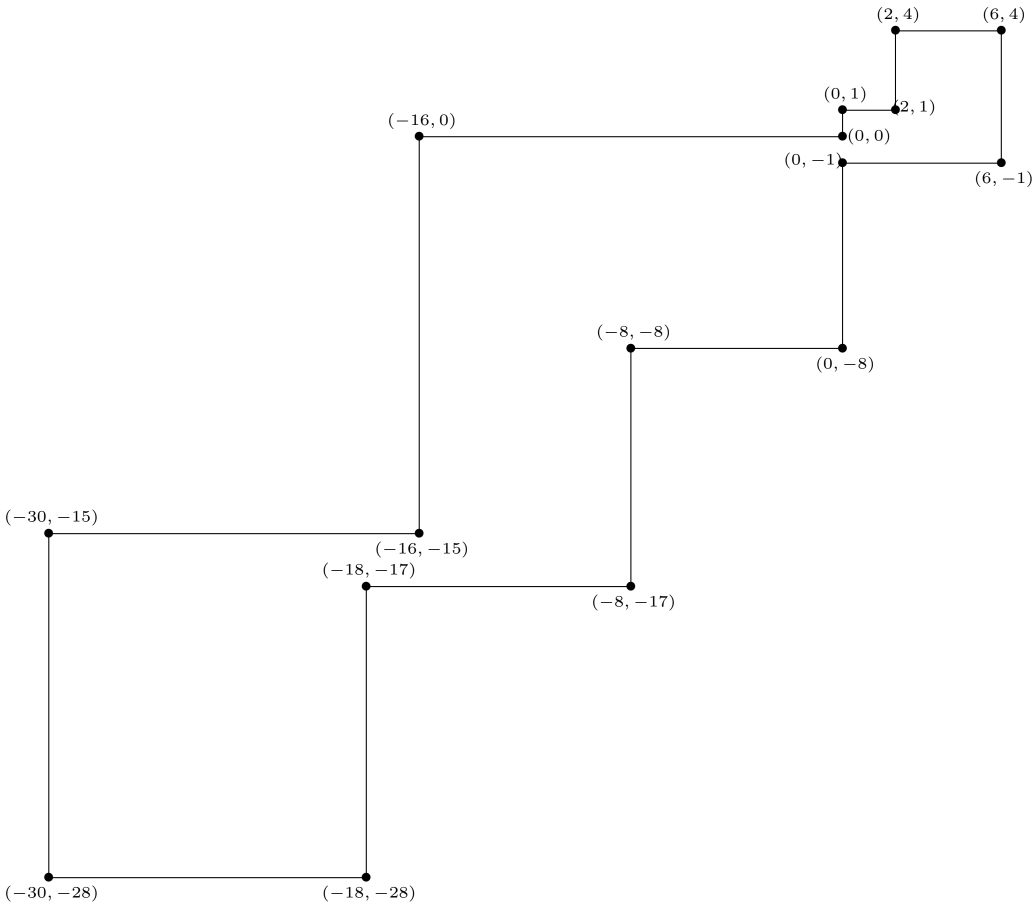
Solution: Since $S(n) \leq 9 \log n$, it follows that n must have at least 200 digits. In fact if n has few than 220 digits, then $S(n) \leq 219 \cdot 9 = 1971$ and $S(S(n)) \leq 1 + 9 + 6 + 9 = 25$ and $s(S(S(n))) \leq 10$, in which case their sum is at most 2006. Since $n \equiv S(n) (\text{mod } 9)$, and $2009 \equiv 0 (\text{mod } 9)$, it follows that any solution n satisfies one of $n \equiv 0 (\text{mod } 9)$, $n \equiv 3 (\text{mod } 9)$, or $n \equiv 6 (\text{mod } 9)$. We are left to try multiples of 3 whose sum of digits is at least 1971. Trying 1971, 1974, 1977, 1980, 1983, etc. We see that 1977 works. The other $S(n)$ values that work are 1980, 1983, and 2001. The smallest number for which $S(n) = 1977$ is a digit 6 followed by a sting of two-hundred and nineteen 9's. That is, $6 \cdot 10^{219} + 10^{219} - 1 = 7 \cdot 10^{219} - 1$.

10.3 Let S and T be finite disjoint subsets of the plane. Prove that there exists a family L of parallel lines such that each point of S belongs to a member of L and no member of T belongs to any member of L.

Solution: Solution. Let R denote the collection of slopes of lines joining the pairs $(P, Q), P \in S, Q \in T$. (This may include ∞.) Next pick any real number r not in R. The family of lines with slope r through points of S satisfies the required conditions.

10.4 A bug starts from the origin on the plane and crawls one unit upwards to $(0, 1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2, 1)$. Then during the third minute, it crawls three units upward, arriving at $(2, 4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then making a $90°$ turn either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the smallest possible area of the 16-gon traced out by its path?

Solution: The answer is 384. Note that the vertical sides of the polygon are all odd lengths. Since the up sides equals the down sides, we must partition the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ into two subsets, one containing both 1 and 3 so that the sum of the members of each subset is $(1 + 3 + 5 + 7 + 9 + 11 + 13 + 15) \div 2 = 32$. Experimentation shows that this can be done in only one way: $Up = \{1, 3, 13, 15\}$ and $Down = \{5, 7, 9, 11\}$. The horizontal edges are trickier. The sum $2 + 4 + 6 + \cdots + 16 = 72$ so the partition must be into two sets each with sum 36 such that the 'right' edges include both 2 and 4. This can be done in four ways: $\{2, 4, 14, 16 | 6, 8, 10, 12\}, \{2, 4, 8, 10, 12 | 6, 14, 16\}, \{2, 4, 6, 8, 16 | 10, 12, 14\}, \{2, 4, 6, 10, 14 | 8, 12, 16\}$. Only two of these give rise to non-intersecting paths. The path for $Right = \{2, 4, 14, 16\}, Left = \{6, 8, 10, 12\}$ is shown below. The other nonintersecting path is given by the partition $Right = \{2, 4, 6, 8, 16\}, Left = \{10, 12, 14\}$, $Up = \{1, 3, 13, 15\}$ and $Down = \{5, 7, 9, 11\}$, which gives rise to a 16-gon whose area is 660.


(2, 4)
(6, 4)
(0, 1)
(2, 1)
(−16, 0)
(0, 0)
(0, −1)
(6, −1)
(−8, −8)
(0, −8)
(−30, −15)
(−16, −15)
(−18, −17)
(−8, −17)
(−30, −28)
(−18, −28)