My Favorite Problems, 10

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This is the tenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

10.1 (This is problem appears in The Art and Craft of Problem Solving (Problem 7.2.13), by Paul Zeitz.) A sequence of integers is defined by $a_0 = p$, where $p > 0$ is a prime number, $a_n + 1 = 2a_n + 1$, for $n = 0, 1, 2, \ldots$. Is there a value of p such that the sequence consists entirely of prime numbers?

10.2 Let $S(n)$ denote the sum of the decimal digits of of the integer n. For example $S(64) = 10$. Find the smallest integer n such that

$$S(n) + S(S(n)) + S(S(S(n))) = 2007.$$

10.3 Let S and T be finite disjoint subsets of the plane. Prove that there exists a family L of parallel lines such that each point of S belongs to a member of L and no member of T belongs to any member of L.

10.4 A bug starts from the origin on the plane and crawls one unit upwards to $(0, 1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2, 1)$. Then during the third minute, it crawls three units upward, arriving at $(2, 4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then making a $90°$ turn either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the smallest possible area of the 16-gon traced out by its path?

Problems from My Favorite Problems, 9, with solutions.

9.1 The Infected Checkerboard. This problem appears in the essay Five Algorithmic Puzzles by Peter Winkler in A Tribute to a Mathemagician. He attributed the problem to KVANT around 1986. Given a 10×10 checkerboard of squares, some of which are darkened (contaminated). The infection spreads among the squares as follows: if a square has two or more infected neighbors, then it becomes infected. Neighboring means sharing an edge. If we start with all the squares on the main diagonal infected, clearly all the squares will eventually become infected. Prove that no fewer than 10 squares can accomplish this complete contamination.

Solution: The solution can be summarized in one word: perimeter! Look at a few cases and you'll see that the perimeter of the set of infected squares never increases. Thus, starting with at most nine squares (perimeter 36), we can never achieve a position for which the perimeter of the infected squares is 40.

9.2 Two perfect logicians, S and P, are told that integers x and y have been chosen such that $1 < x < y$ and $x + y < 100$. S is given the value $x + y$ and P is given the value xy. They then have the following conversation. P: I cannot determine the two numbers. S: I knew that. P: Now I can determine them. S: So can I. Given that the above statements are true, what are the two numbers? This is problem 3 at Nick Hobson's website Nick's Mathematical Puzzles `http://www.qbyte.org/puzzles/p003s.html` Also, see `http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/logic_sum_product`

Solution: First of all, trivially, xy cannot be prime. It also cannot be the square of a prime, for that would imply $x = y$. We now deduce as much as possible from each of the logicians' statements. We have only public information: the problem statement, the logicians' statements, and the knowledge that the logicians, being perfect, will always make correct and complete deductions. Each logician has, in addition, one piece of private information: sum or product.

P: I cannot determine the two numbers. P's statement implies that xy cannot have exactly two distinct proper factors whose sum is less than 100. Call such a pair of factors *eligible*. For example, xy cannot be the product of two distinct primes, for then P could deduce the numbers. Likewise, xy cannot be the cube of a prime, such as $3^3 = 27$, for then $3 \cdot 9$ would be a unique factorization; or the fourth power of a prime.

Other combinations are ruled out by the fact that the sum of the two factors must be less than 100. For example, xy cannot be $242 = 2 \cdot 112$, since 1122 is the unique eligible factorization; $2 \cdot 121$ being ineligible. Similarly for $xy = 318 = 2353$.

S: I knew that. If S was sure that P could not deduce the numbers, then none of the possible summands of $x + y$ can be such that their product has exactly one pair of eligible factors. For example, $x + y$ could not be 51, since summands 17 and 34 produce $xy = 578$, which would permit P to deduce the numbers.

We can generate a list of values of $x+y$ that are never the sum of precisely two eligible factors. The following list is generated by JavaScript. Eligible sums: $11, 17, 23, 27, 29, 35, 37, 41, 47, 53$.

(We can use Goldbach's Conjecture, which states that every even integer greater than 2 can be expressed as the sum of two primes, to deduce that the above list can contain only odd numbers. Although the conjecture remains unproven, it has been confirmed empirically up to $4 \cdot 10^{14}$.)

P: Now I can determine them.

P now knows that $x + y$ is one of the values listed above. If this enables P to deduce x and y, then, of the eligible factorizations of xy, there must be precisely one for which the sum of the factors is in the list. The table at `http://www.math.uncc.edu/~hbreiter/problems/M&IQ/M&IQ10.pdf`, generated by JavaScript (view plain text JavaScript: function genProd), shows all such xy, together with the corresponding x, y, and $x + y$. The table is sorted by sum and then product.

Note that a product may be absent from the table for one of two reasons. Either none of its eligible factorizations appears in the above list of eligible sums (example: $12 = 2 \cdot 6$ and $3 \cdot 4$; sums 8 and 7), or more than one such factorization appears (example: $30 = 2 \cdot 15$ and 56; sums 17 and 11.)

S: So can I. If S can deduce the numbers from the table below, there must be a sum that appears exactly once in the table. Checking the table, we find just one such sum: 17. Therefore, we are able to deduce that the numbers are $x = 4$ and $y = 13$.

9.3 Find all integers $1 \leq k \leq 169$ for which 169 is the sum of k nonzero squares. The squares are not necessarily unique. For example $k = 5$: 169=1+4+4+16+144. Solution by Todd Lee, Elon College, Elon, NC.

Solution: First Wave: For $k = 169$, we have the sum of 169 1's. We need to trade 4 of the 1's to get the next square sum, $165 \cdot 1 + 4$, so there are square sums with 168 and 167 terms. By the same argument, $161 \cdot 1 + 2 \cdot 4$ skips over square sums with 165 and 164 terms. With this technique of substituting 4 for 4 ones covers the k's: $k = 169, 166, 163, 160, ..., 431 + 42 \cdot 4$.

Once down to $k = 163$ where we have at least two 4's, Substituting 9 for $1 + 2 \cdot 4$ on each of these give a net loss of two on k: $k = 161, 158, 155, 152, \ldots, 41 \rightarrow 40 \cdot 4 + 9$. Starting with $k = 155$, we can do another substitution on 9 for $1 + 2 \cdot 4$ on the last set to cover: $k = 153, 150, 147, 144, \ldots, 42 \rightarrow 3 \cdot 1 + 37 \cdot 4 + 2 \cdot 9$.

This spread does devastating damage once we get started. The numbers $168, 167, 165, 164, 162, 159, 152$ are the only $k > 40$ where 169 can not be written k terms of squares.

Second Wave:

Now start with the sum $69 \cdot 1 + 100$ and begin the above game again. By doing just the 4's, we cover: $k = 70, 67, 64, 61, \ldots, 19 \rightarrow 1 + 17 \cdot 4 + 100$.

Starting with $k = 64$, we do the 9 game: $k = 62, 59, 56, 53, \ldots, 17 \rightarrow 15 \cdot 4 + 9 + 100$. Starting with $k = 56$, we do the second 9: $k = 54, 51, 48, 45, \ldots, 18 \rightarrow 3 \cdot 1 + 12 \cdot 4 + 2 \cdot 9 + 100$. Notice that our last gap in this wave is at 52, well within the coverage of the first wave. So, except for the k mentioned in the first wave, we have all k covered for $k > 16$.

Third Wave:

Starting with the sum $48 \cdot 1 + 121$, we can use the same constructions in the first two waves. The last gap is at $k = 31$, and covered by the second wave. The last k covered in this wave is $k = 12$ given by the sum $2 \cdot 1 + 7 \cdot 4 + 2 \cdot 19 + 121$.

Last Wave: $k = 1 \rightarrow 169$;
$k = 2 \rightarrow 25 + 144$;
$k = 3 \rightarrow 9 + 16 + 144$;
$k = 4 \rightarrow 3 \cdot 16 + 121$;
$k = 5 \rightarrow 1 + 2 \cdot 16 + 36 + 100$;
$k = 6 \rightarrow 4 \cdot 4 + 9 + 144$;
$k = 7 \rightarrow 3 \cdot 1 + 4 + 2 \cdot 9 + 144$;
$k = 8 \rightarrow 3 \cdot 1 + 6 \cdot 4 + 121$;
$k = 9 \rightarrow 4 \cdot 1 + 3 \cdot 4 + 9 + 144$;
$k = 10 \rightarrow 4 \cdot 1 + 3 \cdot 4 + 2 \cdot 16 + 121$;
$k = 11 \rightarrow 3 \cdot 1 + 4 \cdot 4 + 9 + 16 + 25 + 100$;

Conclusion: We can write 169 as the sum of k square terms, for $k < 170$, k not in the set $\{168, 167, 165, 164, 162, 159, 156\}$.