Problem Set 2, Handbook 2000 Problems

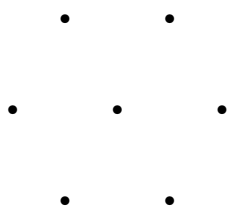
1. How many integers in the set $\{1, 2, 3, 4, \ldots, 360\}$ have at least one prime divisor in common with 360?

2. Use the digits 2 through 9, one per square, to maximize the value of

$$\square\square\square\square + (\square\square \times \square\square).$$

   What is that maximum value?

3. How many equilateral triangles have all three vertices in the hexagonal lattice shown?

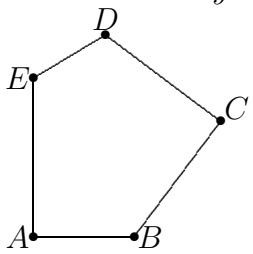


4. The product of the digits of a four-digit number is 6!. What is the largest the number could be? What is the smallest it could be? How many such numbers are there?

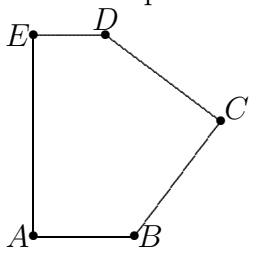
5. Find all pairs of positive integers, x, y such that

$$1 + 4x + 6y = xy.$$

6. The pentagon $ABCDE$ with vertices $A = (0, 0)$, $B = (7, 0)$, $C = (13, 8)$, $D = (5, 14)$, and $E = (0, y)$ has perimeter 41. What is y?



7. The pentagon P with vertices $A = (0, 0)$, $B = (7, 0)$, $C = (13, 8)$, $D = (5, 14)$, and $E = (0, 14)$. A line L through the origin divides P into two pentagons with equal perimeters. Find the coordinates of the point F where L meets $\overline{CD}$.



8. There are eight unit squares that have two or more vertices in the 2 by 3 array of lattice points $\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{matrix}$
How many unit squares have at least two vertices in an m by n array of lattice points?

9. Each of the six faces of a plastic cube is colored either red or green with equal probability. What is the probability that such a coloring results in a cube that has a vertex, all three of whose containing faces is the same color?

10. Three faces of a cube are randomly selected. What is the probability that they have a common vertex?

11. Find a number that differs by 1 from the sum of the squares of its digits.

12. A point is randomly selected from the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,3)$. What is the probability that the point is within one unit of $(0,0)$? Express your answer in terms of π.

13. A point is randomly selected from the rectangle with vertices at $(0,0)$, $(2,0)$, $(2,3)$ and $(0,3)$. What is the probability that the x-coordinate of the point is less than the y-coordinate?

14. Both $\odot$ and $*$ are in the set $\{+, \times, \div, -\}$, and

$$(12 \odot 2) \div (9 * 3) = 2/9.$$

Compute the value of $(8 \odot 4) \div (1 * 2)$.

15. Compute the value of $99^3 + 3 \cdot 99^2 + 3 \cdot 99$

16. There are several sets of three different numbers whose sum is 14 which can be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many of these sets contain a 4?

17. Two circles of radius 1 are centered at $(4,0)$ and $(-4,0)$. How many circles contain exactly one point of each of the given circles and also the point $(0,5)$?

18. How many integers in the range 500 to 999 have no consecutive identical digits. For example, 626 qualifies but 722 does not.

19. The function f is linear and satisfies $f(d+1) - f(d) = 3$ for all real numbers d. What is $f(3) - f(5)$?

Answers

1. 264
2. $15,932 = 9632 + 84 \times 75$
3. 8
4. $9852, 2589, 72$
5. Transform to $xy-6y-4x-5^2+4\cdot 6 = 0$. Then factor to get $(x-6)(y-4) = 25$. Finally, find six different pairs.
6. 1
7. $6/10(5, 14) + 4/10(13, 8) = (8.20, 11.60)$
8. $mn + 2m + 2n$
9. $46/64 = 23/32$
10. $2/5$
11. $35, 75$
12. $\pi/12$
13. $1/3$
14. 1
15. $100^3 - 1 = 999,999$
16. 3
17. 4
18. 405
19. -6