UNC Charlotte   2014   Comprehensive Exam

March 3, 2014

1. The origin and the points where the line ℓ intersects the x-axis and the y-axis are the vertices of a right triangle T whose area is 5. Also the line ℓ is perpendicular to the line given by the equation $5x - y = 15$. What is the length of the hypotenuse of T?

(A) $\sqrt{20}$ (B) $\sqrt{26}$ (C) $\sqrt{29}$ (D) $\sqrt{45}$ (E) $\sqrt{52}$

2. A two-digit number is written at random. What is the probability that the sum of its digits is 5?

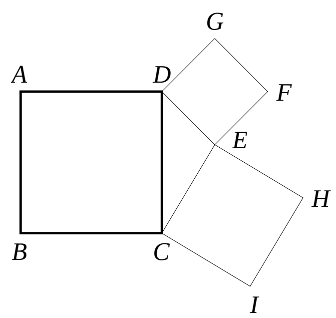
(A) $\frac{5}{89}$ (B) $\frac{1}{18}$ (C) $\frac{6}{89}$ (D) $\frac{1}{15}$ (E) $\frac{4}{89}$

3. Two people toss fair coin. One threw it 10 times while the other – 11 times. What is the probability that the second coin's tail appeared more times than the first?

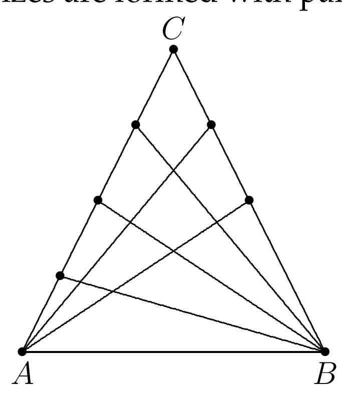
(A) $\frac{511}{1024}$ (B) $\frac{513}{1024}$ (C) $\frac{1025}{2028}$ (D) $\frac{1027}{2048}$ (E) $\frac{1}{2}$

4. Three squares are arranged as in the diagram. The largest ($\square$ABCD) has area 36 and the smallest ($\square$DEFG) has area 8. If triangle $\triangle$CDE has area 6, what is the area of the other square ($\square$CEHI)?

(A) 18 (B) 20 (C) 22 (D) 24 (E) 28



5. The figure below is built with eight line segments, each with either A or B as an endpoint. Triangles of various sizes are formed with parts of these segments as boundary.



How many triangular regions are there?

(A) 19 (B) 32 (C) 36 (D) 38 (E) 42

6. Cucumbers contain 99% of water. After being exposed to the sun the amount of water drops to 98%. What percentage of weight did the cucumbers lose?

(A) 0.98% (B) 1% (C) 2% (D) 4% (E) 50%

7. Suppose a is a real number for which $a^2 + \frac{1}{a^2} = 14$. What is the largest possible value of $a^3 + \frac{1}{a^3}$?

(A) 48 (B) 52 (C) 56 (D) 60 (E) 64

8. Let $p(x) = (x-7)(x^3+5x^2+7x-11) + (x-9)(x^3+5x^2+7x-11)$. What is the sum of the roots of $p(x) = 0$?

(A) -11 (B) -3 (C) 3 (D) 11 (E) 13

9. Suppose a, b, c are integers satisfying $1 \leqslant a < b < c$ and $a^2 + b^2 + c^2 = 14(a+b+c)$. What is the sum $a+b+c$?

(A) 38 (B) 39 (C) 40 (D) 41 (E) 42

10. What is the largest prime divisor of $2^{16} - 16$?

(A) 7 (B) 13 (C) 17 (D) 19 (E) 23

11. Suppose the roots of the equation $(x^2 - 2x + m)(x^2 - 2x + n) = 0$, where m, n are two real numbers, form an arithmetic sequence with the first term being $\dfrac{1}{4}$. Then $|m - n| =$

(A) 1 (B) $\dfrac{3}{4}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{8}$ (E) 2

12. If $\sin x + \cos x = 1.2$, then what is the value of $\sin 2x$?

(A) 0.22 (B) 0.88 (C) -0.2 (D) 0.44 (E) -0.88

13. Let $i = \sqrt{-1}$. Then $(\sin 6^\circ + i \cos 6^\circ)^{30} =$

(A) 1 (B) i (C) -1 (D) $\dfrac{\sqrt{2}}{2}$ (E) $-\dfrac{\sqrt{2}}{2}$

14. The sides of a right triangle are a, $2a + 2d$ and $2a + 3d$, with a and d both positive. The ratio of a to d is:

(A) $5:1$ (B) $27:2$ (C) $4:1$ (D) $1:5$ (E) $2:3$

15. Two armies are advancing towards each other, each one at 1 mph. A messenger leaves the first army when the two armies are 10 miles apart and runs towards the second at 9 mph. Upon reaching the second army, he immediately turns around and runs towards the first army at 9 mph. How many miles apart are the two armies when the messenger gets back to first army?

(A) 5.6 (B) 5.8 (C) 6 (D) 6.2 (E) 6.4

16. A machine was programmed to transmit a certain sequence of five digits, all zeros and ones, five times. One time it did it correctly; one time it did so with one mistake; one time it did so with two mistakes; one time it did so with three mistakes; one time it did so with four mistakes. The five transmissions are listed below. Which is the correct sequence?

(A) 00001 (B) 00100 (C) 01100 (D) 10010 (E) 10011

17. Oil is pumped into a non-empty tank at a changing rate. The volume of oil in the tank doubles every minute and the tank is filled in 10 minutes. How many minutes did it take for the tank to be half full?

(A) 2 (B) 5 (C) 7 (D) 8 (E) 9

18. Jack and Lee walk around a circular track. It takes Jack and Lee respectively 6 and 10 minutes to finish each lap. They start at the same time, at the same point on the track, and walk in the same direction around the track. After how many minutes will they be at the same spot again (not necessarily at the starting point) for the first time after they start walking?

(A) 15 (B) 16 (C) 30 (D) 32 (E) 60

19. Imagine that Rubik's cube consists of 27 equal cubes. Find the minimal amount of hits of an axe that is needed in order to divide Rubik's cube into 27 equal cubes?

(A) 6 (B) 8 (C) 10 (D) 12 (E) 16

20. A triangular pyramid is placed inside a sphere so that it does not intersect or touches the sphere. Each of the faces of the pyramid is extended so that they become planes. In how many pieces will the extensions cut the sphere?

(A) 9 (B) 10 (C) 12 (D) 14 (E) 15

21. Let $a_n = \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$. Find the sum $a_1 + a_2 + \cdots + a_{99}$.

(A) 6 (B) 8 (C) 9 (D) 12 (E) 15

22. Evaluate exactly $\sqrt{5+2\sqrt{6}} + \sqrt{5-2\sqrt{6}}$.

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $2\sqrt{3}$ (D) $2\sqrt{5}$ (E) $\frac{3}{2}\sqrt{6}$

23. What is the largest power of 2 that divides the number $K = 75! - 71!$?

(A) 2^{62} (B) 2^{63} (C) 2^{65} (D) 2^{66} (E) 2^{67}

24. Consider the sequence $a_1 = 1, a_2 = 13, \ldots$ where each term a_n is obtained from the previous term a_{n-1} by appending the n^{th} odd number. So $a_3 = 135, a_4 = 1357$, etc. Find the number m so that a_m is the 30^{th} multiple of 9 in the sequence.

(A) 66 (B) 77 (C) 81 (D) 90 (E) 99

25. Let P denote the point on the circle $x^2 + 2x + y^2 - 4y = 20$ that is closest to $(7, 8)$. What is the slope of the line passing through both P and $(7, 8)$?

(A) 2/3 (B) 3/4 (C) 4/5 (D) 1 (E) 2