UNC Charlotte 2008 Algebra

March 3, 2008

1. The sum of all divisors of 2008 is

   (A) 8 **(B)** 1771 **(C)** 1772 **(D)** 3765 **(E)** 3780

   Solution: E. The prime factorization is $2008 = 2^3 \cdot 251$, which has the divisors $1, 2, 4, 8$ and $1 \cdot 251, 2 \cdot 251, 4 \cdot 251, 8 \cdot 151$. Their sum is $15 \cdot 252 = 3780$.

2. From the list of all natural numbers $2, 3, \ldots 999$, delete nine sublists as follows. First, delete all even numbers except 2, then all multiples of 3 except 3, then all multiples of 5 except 5, and so on, for the nine primes $2, 3, 5, 7, 11, 13, 17, 19, 23$. Find the sum of the **composite** numbers left is the remaining list.

   (A) 0 **(B)** 899 **(C)** 961 **(D)** 2701 **(E)** 3062

   Solution: D. The composite numbers left have all prime factors at least 29. There are three of them less than 1000: $29 \cdot 29 = 841$, $29 \cdot 31 = 899$ and $31 \cdot 31 = 961$, with the sum $841 + 899 + 961 = 2701$.

3. The polynomial $P(x) = (x^6 - 1)(x - 1) - (x^3 - 1)(x^2 - 1)$ has potentially 7 real zeros. Which of the following is a zero of multiplicity greater than 1?

   (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

   Solution: D. One can factor the polynomial as $(x^6 - 1)(x - 1) - (x^3 - 1)(x^2 - 1) = (x^3 - 1)(x^3 + 1)(x - 1) - (x^3 - 1)(x - 1)(x + 1) = (x^3 - 1)(x - 1)[(x^3 + 1) - (x + 1)] = (x - 1)(x^2 + x + 1)(x - 1)x(x - 1)(x + 1) = x(x - 1)^3(x + 1)(x^2 + x + 1)$ Hence $x = 1$ is the only multiple zero.

4. New York City and Washington D.C. are about 240 miles apart. A car leaves New York City at noon traveling directly south toward Washington D.C. at 55 miles per hour. At the same time and along the same route, a second car leaves Washington D.C. bound for New York City traveling directly north at 45 miles per hour. How far has the car which left New York City traveled when the drivers meet for lunch at 2:24 P.M.?

   (A) 128 miles **(B)** 130 miles **(C)** 131 miles **(D)** 132 miles **(E)** 134 miles

   Solution: D. When the two cars meet they will have traveled a total of 240 miles in the time $240 \div (55 + 45) = 2.4$ hours. The faster car travels $55 \times 2.4 = 132$ miles during this time.

5. Suppose $x + 1/y = 1.5$ and $y + 1/x = 3$. What is $x \div y$?

   (A) 0.2 **(B)** 0.3 **(C)** 0.4 **(D)** 0.5 **(E)** 1.2

   Solution: D. We have $xy + 1 = 1.5y$ and $xy + 1 = 3x$. So, $y = 2x$. It follows from this that $x \div y = 0.5$.

6. If $x + y + z = 7$ and $x^2 + y^2 + z^2 = 21$, what is $xy + yz + zx$.

7. During a rebuilding project by contractors A, B, and C, there was a shortage of tractors. The contractors lent each other tractors as needed. At first, A lent B and C as many tractors as they each already had. A few months later, B lent A and C as many as they each already had. Still later, C lent A and B as many as they each already had. By then each contractor had 24 tractors. How many tractors did contractor A originally have?

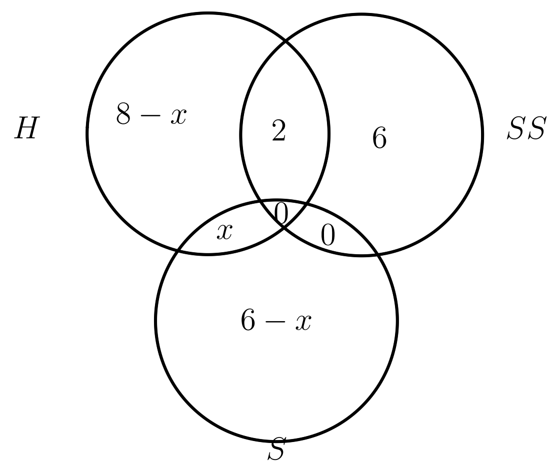
   (A) 21 **(B)** 24 **(C)** 30 **(D)** 33 **(E)** 39

Solution: E. A had 39, B had 21, and C had 12 originally. To see this first note that there are 72 tractors at the end. Letting a, b and c denote the number of tractors each of A, B and C had originally, respectively, it follows that $a+b+c = 72$. After the first exchange, A had $a-b-c$ tractors, and after the second, he has $2(a-b-c)$, then after the third, he had $4(a-b-c)$, which we are told, is 24. This means $a-b-c = 6$. Together with $a+b+c = 72$, it follows that $2a = 78$ and $a = 39$. From this it is not hard to see that $b = 21$ and $c = 12$.

8. In a school of 20 teachers, 10 teach Humanities, 8 teach Social Studies and 6 teach Sciences; 2 teach Humanities and Social Studies, but none teach Social Studies and Sciences. How many teach Humanities and Sciences?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

   Solution: A. Let H, SS, and S denote the sets of teachers of Humanities, Social Science and Science, respectively, as shown below. Letting x denote the number of teachers of both Humanities and Science, it follows that $10 + 8 + 6 - 2 - x = 20$ from which it follows that $x = 2$.



9. A certain integer N has exactly eight factors, counting itself and 1. The numbers 35 and 77 are two of the factors. What is the sum of the digits of N?

   (A) 9 **(B)** 10 **(C)** 16 **(D)** 18 **(E)** 20

   Solution: C. The divisors include $5, 7$, and 11. But $5 \cdot 7 \cdot 11 = 385$ already has 8 divisors.

10. Suppose that x and y are positive real numbers that satisfy the equations $x^2 + xy + y^2 = 7$ and $3x + y = 3$. Find $y - 4x$.

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 3/2 **(E)** 5/2

    Solution: B. Solving the second equation for y, we get $y = 3 - 3x$. Substituting this result into the first equation, we find $x^2 + x(3 - 3x) + (3 - 3x)^2 = 7$. This is equivalent to $7x^2 - 15x + 2 = (7x - 1)(x - 2) = 0$. Thus, $x = 2$ or $x = 1/7$. Since $y = 3 - 3x$, we obtain $y = -3$ or $y = 18/7$ respectively.

    Because $x, y \geq 0$, we conclude $x = 1/7$ and $y = 18/7$. Therefor $y - 4x = 2$.

11. Suppose a, b, and c are integers satisfying

$$\begin{aligned} a + b^2 + 2ac &= 22 \\ b + c^2 + 2ab &= 36 \\ c + a^2 + 2bc &= -2 \end{aligned}$$

    What is $a + b + c$?

    (A) -6 **(B)** -2 **(C)** 4 **(D)** 7 **(E)** 9

    Solution: D. Add the three equations and note that $a + b + c + (a + b + c)^2 = 22 + 36 - 2 = 56$, so $a + b + c$ must be 7.

12. Two points A and B are 4 units apart are given in the plane. How many lines in the plane containing A and B are 2 units from A and 3 units from B?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

    Solution: C. In order that a line be 2 units from A, it must be tangent to the circle of radius 2 about the point A and similarly for B. The circles of radius 2 about A and radius 3 about B intersect, so there are just two lines tangent to both circles.

13. The number of real solutions of the equation $|x-2|+|x-3|=3$ is

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** *many*

    Solution: B. There are just two solutions, $x=1$ and $x=4$. We'll condition on three inequalities. If $x<2$, then $|x-2|=2-x$ and $|x-3|=3-x$, in which case we have $2-x+3-x=3$ or $-2x=-2$, so $x=1$. If $2\leq x\leq 3$, then $|x-2|=x-2$ and $|x-3|=3-x$, in which case we have $x-2+3-x=3$, which is impossible. Finally, if $3<x$, then $|x-2|=x-2$ and $|x-3|=x-3$, in which case we have $x-2+x-3=3$ or $2x=8$, so $x=4$.

14. The numbers $1\leq a,b,c,d,e\leq 2008$ are randomly chosen integers (repetition is allowed). What is the probability that $abc+de$ is even?

    (A) 1/2 **(B)** 1/4 **(C)** 11/16 **(D)** 7/16 **(E)** 21/32

    Solution: C. Each of a,b,c,d,e is even or odd with probability 1/2. The expression $abc+de$ is even if either both of abc and de are odd, or both are even. The probability that abc is odd is 1/8, the probability that de is odd is 1/4. The probability that abc is even is 7/8, the probability that de is even is 3/4. So the probability that $abc+de$ is even is $1/8\times 1/4+7/8\times 3/4=22/32=11/16$.

15. On a fence are sparrows and pigeons. When five sparrows leave, there remain two pigeons for every sparrow. After that twenty-five pigeons leave, and the ratio of sparrows to pigeons becomes three to one. Find the original number of birds.

    (A) 44 **(B)** 48 **(C)** 50 **(D)** 54 **(E)** 60

    Solution: C. Let p denote the number of pigeons and s the number of sparrows. Then $p/(s-5)=2$ and $(s-5)/(p-25)=3$. Solve these two equations simultaneously to get $s=20$ sparrows and $p=30$ pigeons, so the original number of birds is 50.

16. Suppose a and b are digits satisfying $1 < a < b < 8$. Also, the sum $1111 + 111a + 111b + \cdots$ of the smallest eight four-digit numbers that use only the digits $\{1, a, b, 8\}$ is 8994. What is $a + b$?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

    Solution: C. The eight smallest numbers are $1111, 111a, 111b, 1118, 11a1, 11aa, 11ab$ and $11a8$. Their sum is $8858 + 42a + 2b = 8994$, so $42a + 2b = 136$ from which it follows that $21a + b = 68$. From this we can reason that a must be 3, and that b must be 5, so $a + b = 8$.

17. At one of mayor Pat McCrory's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

    (A) 78 **(B)** 185 **(C)** 234 **(D)** 312 **(E)** 325

    Solution: C. Each man shakes hands with all women but his wife, which gives 13*12 handshakes. There are also $\binom{13}{2} = (13 * 12)/2$ handshakes between men; totally, there are $(13 * 12) * (3/2) = 234$ handshakes.

18. On a die, 1 and 6, 2 and 5, 3 and 4 appear on opposite faces. When 2 dice are thrown, multiply the numbers appearing on the top and bottom faces of the dice as follows:

    (a) number on top face of 1st die $\times$ number on top face of 2nd die

    (b) number on top face of 1st die $\times$ number on bottom face of 2nd die

    (c) number on bottom face of 1st die $\times$ number on top face of 2nd die

    (d) number on bottom face of 1st die number $\times$ on bottom face of 2nd die.

What can be said about the sum S of these 4 products?

(A) The value of S depends on luck and its expected value is 48

(B) The value of S depends on luck and its expected value is 49

(C) The value of S depends on luck and its expected value is 50

(D) The value of S is 49

(E) The value of S is 50

Solution: D. Suppose U and B are the up and bottom on the first die and u and b for the second. Then the sum S equals $Uu + Ub + Bu + Bb = (U + B)(u + b) = 7 \cdot 7 = 49$.

19. During recess, one of five pupils wrote something nasty on the chalkboard. When questioned by the class teacher, the following ensured:

    A : It was 'B' or 'C'.

    B : Neither 'E' nor I did it.

    C : You are both lying.

    D : No, either A or B is telling the truth.

    E : No, 'D', that is not true.

    The class teacher knows that three of them never lie while the other two cannot be trusted. Who was the culprit?

    (A) A **(B)** B **(C)** C **(D)** D **(E)** E

    Solution: C. If D's statement is false, then both A and B are also lying, which would mean that we have three liars, and that is impossible. So D's statement is true and therefore E's statement is false. Since C's statement is also false, it must be that A, B and D are honest. But A says it was B or C and B denies it, so only C is left.

20. How many distinct real number solutions does $(3x^2 + 2x)^2 = (x^2 + 2x + 1)^2$ have?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

    Solution: D. Using the formula $a^2 - b^2 = (a - b)(a + b)$ we obtain:

    $$(3x^2 + 2x)^2 - (x^2 + 2x + 1)^2 = (2x^2 - 1)(4x^2 + 4x + 1) = (2x^2 - 1)(2x + 1)^2,$$

    so the only zeros of this polynomial are $\pm 1/\sqrt{2}$ and -1. So there are a total of 3 real solutions.

21. Let N be the largest 7-digit number that can be constructed using each of the digits $1, 2, 3, 4, 5, 6,$ and 7 such that the sum of each two consecutive digits is a prime number. What is the reminder when N is divided by 7?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

    Solution: E. The number is $N = 7652341$, which is constructed from left to right. The reminder is 4.

22. For how many n in $\{1, 2, 3, \ldots, 100\}$ is the tens digit of n^2 odd?

    (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

    Solution: E. There are 2 in each decile, $10a + 4$ and $10a + 6$. The tens digit of $(10a + 4)^2 = 100a^2 + 80a + 16$ is the units digit of $8a + 1$, while the tens digit of $(10a + 6)^2 = 100a^2 + 120a + 36$ is the units digit of $2a + 3$, both of which are odd for any integer a. All the other tens digits of perfect squares are even: $(10a + b)^2 = 100a^2 + 20a + b^2$, the tens digit of which is the tens digit of $2a + b^2$, which is even if the tens digit of b^2 is even. But the tens digit of b^2 is even if $b \neq 4, b \neq 6$.

23. How many pairs of positive integers (a, b) with $a + b \leq 100$ satisfy

$$\frac{a + b^{-1}}{a^{-1} + b} = 13?$$

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

Solution: E. Multiplying the given equality $a + b^{-1} = 13(b + a^{-1})$ by ab we obtain: $a(ab+1) = 13b(ab+1)$, or $(a-13b)(ab+1) = 0$. Since $ab+1 > 0$, the given equation is equivalent to $a = 13b$. The inequality $a+b \leq 100$ means that $14b \leq 100$; therefore, the possible values of the positive integer b are $1, 2, \ldots, 7$, and there are 7 solutions: $(13, 1)$, $(26, 2), \ldots, (91, 7)$.

24. The numbers $1, 2, 4, 8, 16, 32$ are arranged in a multiplication table, with three along the top and the other three down the column. The multiplication table is completed and the sum of the nine entries is tabulated. What is the largest possible sum obtainable.

(A) 902 **(B)** 940 **(C)** 950 **(D)** 980 **(E)** 986

$\times$	a	b	c
d			
e			
f			

Solution: D. The sum is $(a+b+c) \cdot (d+e+f)$ which is as large as possible when the two factors $a+b+c$ and $c+d+e$ are as close together as possible (given that the sum is constant(63)). We must pair the 1 and the 2 with 32 to get $35 \cdot 28 = 980$.