UNC Charlotte 2014 Level 2 Exam

March 3, 2014

1. The origin and the points where the line ℓ intersects the x-axis and the y-axis are the vertices of a right triangle T whose area is 5. Also the line ℓ is perpendicular to the line given by the equation $5x - y = 15$. What is the length of the hypotenuse of T?

(A) $\sqrt{20}$ (B) $\sqrt{26}$ (C) $\sqrt{29}$ (D) $\sqrt{45}$ (E) $\sqrt{52}$

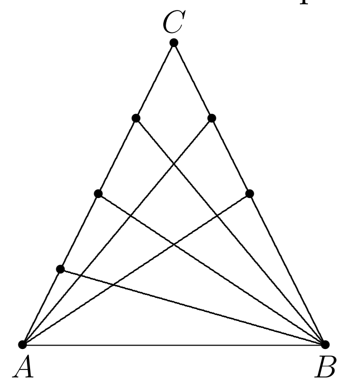
Solution: E. Let $(f, 0)$ be the x-intercept of ℓ and let $(0, g)$ be the y-intercept of ℓ. Then the area of T is $|fg/2|$ and the length of the hypotenuse is $\sqrt{f^2 + g^2}$. There is an equation for ℓ of the form $x + 5y = C$ for some number C with $f = C = 5g$. So $5 = 5g^2/2$ and from this we get $g = \pm\sqrt{2}$ and $f = \pm 5\sqrt{2}$. Thus the hypotenuse has length $\sqrt{52}$.

2. An army company is marching along at 3 mph down a straight road that is exactly 88 feet wide. The company is marching in a rectangular formation that is as wide as the road. Captain Allen likes to march at 5 mph, keeping just ahead of his company by zigzagging across the entire width of the road at the precise angle to stay exactly 2 yards in front of his troops at all times. If the company marches for 6 miles, approximately how many times will Captain Allen have gone from one side of the road to the other?

(A) 390 (B) 420 (C) 480 (D) 540 (E) 620

Solution: C. It will take 2 hours for the company to march 6 miles. They will have covered 31,680 feet. So Captain Allen will have marched 10 miles, or 52,800 feet. In crossing the road once, the Captain will be walking along the hypotenuse of a right triangle where one leg is 88 feet long (the one across the road) and the other leg is how far the troops walk in the time it takes for the Captain to cross the road once. Since the company marches at 3 mph and the Captain at 5 mph, the length of the hypotenuse will be 5/3 times the distance his troops march which means the triangle formed is a 3 – 4 – 5 right triangle. Thus his troops go $66 (= 3 \cdot 22)$ feet in the time it takes the Captain to cross over the road once and the Captain will go $110 (= 5 \cdot 22)$ feet. Therefore Captain Allen will cross from one side to the other $31,680/66 = 480 = 52,800/110$ times.

3. The figure below is built with eight line segments, each with either A or B as an end-point. Triangles of various sizes are formed with parts of these segments as boundary.



How many triangular regions are there?

(A) 19 (B) 32 (C) 36 (D) 38 (E) 42

Solution: E. Each triangle contain A or B or both. There are $6 \cdot 4 = 24$ that contain A and $10 \cdot 3 = 30$ that contain B. But we are double counting $4 \cdot 3 = 12$ triangles so there are $24 + 30 - 12 = 42$ triangular regions.

4. An $a \times b \times c,\ 2 \leqslant a \leqslant b \leqslant c$ rectangular block is built from abc unit cubes. From one corner you can see faces of three different sizes. Suppose you can see exactly 36 of the abc cubes. What is $a^2 + b^2 + c^2$?

(A) 41 (B) 50 (C) 55 (D) 60 (E) 65

Solution: B. Noting that the number $ab + ac + bc$ overcounts by $a + b + c - 1$, we have $ab + ac + bc - a - b - c + 1 = 36$. Trying $a = 2$ results in $b + bc + c + 1 = 38$ which has only solution $b = 1$ and $c = 18$, and this is ruled out. In case $a = 3$ we get $2b + bc + 2c = 38$, which we can solve to get $b = 4$ and $c = 5$. So the answer we seek is $a^2 + b^2 + c^2 = 3^2 + 4^2 + 5^2 = 50$.

5. Suppose a is a real number for which $a^2 + \dfrac{1}{a^2} = 14$. What is the largest possible value of $a^3 + \dfrac{1}{a^3}$?

(A) 48 (B) 52 (C) 56 (D) 60 (E) 64

Solution: B. Compute $(a + 1/a)^2$ to get $a^2 + 2 + 1/a^2 = 14 + 2$, so $a + 1/a$ is either 4 or -4. Then $(a + 1/a)^3 = a^3 + 3a + 3/a + 1/a^3 = a^3 + 1/a^3 + 12 = 4^3 = 64$, so $a^3 + \frac{1}{a^3} = 52$.

6. Let $N = 7 + 77 + 707 + 7007 \cdots + 7 \cdot 10^{30} + 7$. When N is written in decimal (base 10) notation, what is the sum of the digits of N?

(A) 207 (B) 209 (C) 214 (D) 217 (E) 220

Solution: E. Our number N is 31 digits long and most of the digits are 7's. But there are 31 sevens to add in the first column, and their sum is 217. So we record the units digit as 7 and carry 21. Thus $N = 777\ldots7987$. The sum of the digits is $29 \cdot 7 + 9 + 8 = 220$.

7. What is the largest power of 2 that divides the number $K = 75! - 71!$?

(A) 2^{62} (B) 2^{63} (C) 2^{65} (D) 2^{66} (E) 2^{67}

Solution: E. We can write $K = 75 \cdot 74 \cdot 73 \cdot 72 \cdot 71! - 71! = 71!(T-1)$, where T is the even number $75 \cdot 74 \cdot 73 \cdot 72$. So the answer we seek is the largest power of 2 that divides $71!$. This is the number $\lfloor \frac{71}{2} \rfloor + \lfloor \frac{71}{4} \rfloor + \lfloor \frac{71}{8} \rfloor + \lfloor \frac{71}{16} \rfloor + \lfloor \frac{71}{32} \rfloor + \lfloor \frac{71}{64} \rfloor = 35 + 17 + 8 + 4 + 2 + 1 = 67$.

8. Let $p(x) = (x-7)(x^3 + 5x^2 + 1x - 11) + (x-9)(x^3 + 5x^2 + 1x - 11)$. What is the sum of the roots of $p(x) = 0$?

(A) -11 (B) -3 (C) 3 (D) 11 (E) 13

Solution: C. Factor $p(x)$ to get $p(x) = (2x - 16)(x^3 - 5x^2 + 7x - 11)$, which has sum of roots $8 - 5 = 3$.

9. Suppose a, b, c are integers satisfying $1 \leqslant a < b < c$ and $a^2 + b^2 + c^2 = 14(a + b + c)$. What is the sum $a + b + c$?

(A) 38 (B) 39 (C) 40 (D) 41 (E) 42

Solution: A. Rewrite the equation as $(a-7)^2 + (b-7)^2 + (c-7)^2 = 147$ after completing the square three times. So we're looking for three squares whose sum is 147. Note that all three are odd or just one is odd. Trying all three odd leads to $11^2 + 5^2 + 1^2 = 121 + 25 + 1 = 147$. This is the only solution.

10. What is the exact value of

$$\frac{\sqrt{13} - \sqrt{11}}{\sqrt{13} + \sqrt{11}} + \frac{\sqrt{13} + \sqrt{11}}{\sqrt{13} - \sqrt{11}}?$$

(A) 22 (B) $3(\sqrt{13} + \sqrt{11})$ (C) 24 (D) 25 (E) 26

Solution: C. Find a common denominator and convert both fractions to their equivalents in that denominator. $\frac{\sqrt{13}-\sqrt{11}}{\sqrt{13}+\sqrt{11}} + \frac{\sqrt{13}+\sqrt{11}}{\sqrt{13}-\sqrt{11}} = \frac{(\sqrt{13}-\sqrt{11})^2 + (\sqrt{13}+\sqrt{11})^2}{13-11} = \frac{24 - 2\sqrt{143} + 24 + 2\sqrt{143}}{2}$ $= 24$.

11. What is the largest prime divisor of $2^{16} - 16$?

(A) 7 (B) 13 (C) 17 (D) 19 (E) 23

Solution: B.

$$2^{16} - 16 = 2^4(2^{12} - 1) = 2^4(2^6 - 1)(2^6 + 1) = 2^4 \cdot 63 \cdot 65 = 2^4 \cdot 9 \cdot 7 \cdot 5 \cdot 13.$$

12. Suppose the roots of the equation $(x^2 - 2x + m)(x^2 - 2x + n) = 0$, where m, n are two real numbers, form an arithmetic sequence with the first term being $\dfrac{1}{4}$. Then $|m - n| =$

(A) 1 (B) $\dfrac{3}{4}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{8}$ (E) 2

Solution: C. Suppose the 4 roots are given by $\dfrac{1}{4}, \dfrac{1}{4} + d, \dfrac{1}{4} + 2d, \dfrac{1}{4} + 3d$, where d is a real number. Further suppose the roots of $x^2 - 2x + m = 0$ are r_1, r_2 and the roots of $x^2 - 2x + n = 0$ are r_3, r_4, then from $r_1 + r_2 + r_3 + r_4 = 2 + 2$, we have $d = \dfrac{1}{2}$. Since $r_1 + r_2 = 2$, $r_3 + r_4 = 2$, and $r_1, r_2, r_3, r_4 \in \left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\right\}$, there are only two possibilities: either $r_1, r_2 \in \{\frac{3}{4}, \frac{5}{4}\}$, $r_3, r_4 \in \{\frac{1}{4}, \frac{7}{4}\}$; or $r_1, r_2 \in \{\frac{1}{4}, \frac{7}{4}\}$, $r_3, r_4 \in \{\frac{3}{4}, \frac{5}{4}\}$. In both two cases, we have $|m - n| = |r_1 \cdot r_2 - r_3 \cdot r_4| = \frac{1}{2}$.

13. If $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$, then $f\left(\dfrac{1}{x}\right)$ is equal to:

(A) $f(x)$ (B) $-f(x)$ (C) $\dfrac{1}{-f(x)}$ (D) $\dfrac{-x^2 - 1}{1 - x^2}$ (E) $\dfrac{1}{x}$

Solution: B. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$, Replace x by $\dfrac{1}{x}$: $f\left(\dfrac{1}{x}\right) = \dfrac{(\frac{1}{x})^2 + 1}{(\frac{1}{x})^2 - 1} = \dfrac{1 + x^2}{1 - x^2}$ Comparing both equations above, $f\left(\dfrac{1}{x}\right) = -f(x)$

14. In the equation

$$\frac{X}{3} + \frac{Y}{4} = \frac{11}{12}$$

X, Y are natural numbers. Find $X + Y$.

(A) $X + Y = 1$ (B) $X + Y = 2$ (C) $X + Y = 3$ (D) $X + Y = 4$ (E) $X + Y = 3$

Solution: C. $X = 2, Y = 1$.

15. Let ABC, DEF be two three digits numbers and different letters represent different digits, and neither the lead digits A and D is zero. Find the maximal difference

$$ABC - DEF$$

(A) 888 (B) 885 (C) 875 (D) 864 (E) 854

Solution: B. $987 - 102 = 885$.

16. How many solutions does the equation $x^2 - [x^2] = \{x\}^2$ have on the interval $[1, 3]$? Here $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts of a number.

(A) 8 (B) 7 (C) 6 (D) 5 (E) 4

Solution: B. Let $x = m+y$ where $m = [x]$. The equation says that $m^2+2my+y^2-[m^2+2my+y^2] = y^2$. Since m^2 is integer, we can claim that $2my - [2my + y^2] = 0$. This is equivalent to the condition that $2my$ is an integer. Therefore $2my = k \in \{0, 1, 2, 3, ...\}$. There are three possible values of m. They are 1, 2, and 3. If $m = 1$ then $y = k/2$ with $k = 0, 1$. If $m = 2$, then $y = k/4$ with $k = 0, 1, 2, 3$. Finally if $m = 3$ then $y = 0$.

17. How many real solutions does the equation $[x^3] + [x^2] + [x] = \{x\} - 1$ have? Here $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts of a number.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: B. The number on the left side of the equation is integer. Hence, the number on the right side should be an integer. That is, $\{x\} = 0$. Therefore, x is an integer, and we can rewrite the equation as $x^3 + x^2 + x = -1$ or $(x + 1)(x^2 + 1) = 0$ or $x = -1$.

18. Let $a_n = \dfrac{1}{\sqrt{n} + \sqrt{n+1}}$. Find the sum $a_1 + a_2 + \cdots + a_{99}$.

(A) 6 (B) 8 (C) 9 (D) 12 (E) 15

Solution: C. It is easy to see that $a_n = \sqrt{n+1} - \sqrt{n}$. Consequently, $\displaystyle\sum_{1}^{99} a_n = \sqrt{100} - 1$.

19. Evaluate exactly $\sqrt{5 + 2\sqrt{6}} + \sqrt{5 - 2\sqrt{6}}$.

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $2\sqrt{3}$ (D) $2\sqrt{5}$ (E) $\frac{3}{2}\sqrt{6}$

Solution: C. Denoting $x = \sqrt{5 + 2\sqrt{6}} + \sqrt{5 - 2\sqrt{6}}$, we have $x^2 = 5 + 2\sqrt{6} + 5 - 2\sqrt{6} + 2\sqrt{5 + 2\sqrt{6}}\sqrt{5 - 2\sqrt{6}} = 10 + 2\sqrt{5^2 + 4 \cdot 6} = 12$. Hence, $x = \sqrt{12} = 2\sqrt{3}$.

20. Let $f(x) = x^2+12x+30$. How many real solutions does the equation $f(f(f(f(f(x))))) = 0$ have?

(A) 0 (B) 1 (C) 2 (D) 3 (E) $\geqslant 4$

Solution: C. Note that $f(x) = (x+6)^2 - 6$. Then one can see that $f(f(f(f(f(x))))) = (x+6)^{32} - 6$. Hence, $x = -6 \pm 6^{1/32}$.

21. The sum of the first ten terms of a certain arithmetic sequence is 530 and the sum of the first twenty terms is 1860. What is the fifth term of the sequence?

(A) 48 (B) 49 (C) 50 (D) 51 (E) 52

Solution: B. Let a be the first term and let $a+b$ be the second. Then the nth term is $a + b(n-1)$. Also the sum of the first n terms is $na + bn(n-1)/2$. So $530 = 10a + 45b$ and $1860 = 20a + 190b$. Solving for a and b yields $a = 17$ and $b = 8$. Thus the fifth term is $17 + 4 \cdot 8 = 49$.

22. The first three terms of a sequence are $2, 6, 60$ and each term afterwards is the product of the two previous terms, Fibonacci style, except we're multiplying. What is the number of decimal digits of the 6^{th} term of the sequence.

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Solution: C. The sixth term in exponential notation is $2^8 3^5 5^3$, which is $1000 \cdot 6^5$, which is readily seen to be a 7 digit number.

23. The sides of a right triangle are a, $2a+2d$ and $2a+3d$, with a and d both positive. The ratio of a to d is:

(A) $5:1$ (B) $27:2$ (C) $4:1$ (D) $1:5$ (E) $2:3$

Solution: A. For the right triangle, the longest side is the hypotenuse, therefore $a^2 + (2a+2d)^2 = (2a+3d)^2,\ a^2 + (4a^2 + 8ad + 4d^2) = 4a^2 + 12ad + 9d^2,\ a^2 - 4ad - 5d^2 = 0,\ (a-5d)(a+d) = 0,\ a = 5d$ or $a = -d$ (dropped since both a and d $>$0). $a : d = 5d : d = 5:1$.

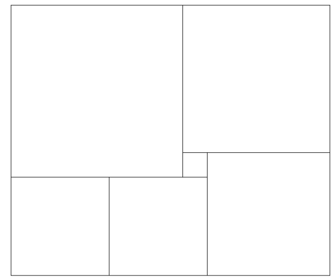
24. Find the value of the product $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{225}\right)$.

(A) $\frac{4}{15}$ (B) $\frac{8}{15}$ (C) $\frac{16}{225}$ (D) $\frac{64}{225}$ (E) $\frac{128}{225}$

Solution: B. The above expression equals $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{15^2}\right)$
$= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\cdots\left(\frac{14}{15}\cdot\frac{16}{15}\right) = \frac{1}{2}\cdot\frac{16}{15} = \frac{8}{15}$.

25. A rectangle consists of six squares. Find the side length of the biggest square if the side of the smallest square is 1.

(A) 5 (B) $\frac{49}{9}$ (C) 6 (D) $\frac{25}{4}$ (E) 7



Solution: E. Denote the side of the biggest square by x. Then the length of the top side of the rectangle is $x + (x - 1)$. The length of the bottom side of the rectangle is $(x - 2) + (x - 3) + (x - 3)$. Since they are equal we get the equation $x + (x - 1) = (x - 2) + (x - 3) + (x - 3)$ or $2x - 1 = 3x - 8$. Hence, $x = 7$.