UNC Charlotte 2014 Level 2 Exam

March 3, 2014

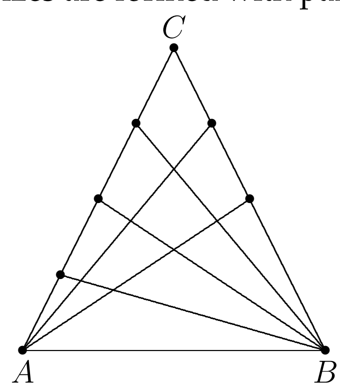
1. The origin and the points where the line ℓ intersects the x-axis and the y-axis are the vertices of a right triangle T whose area is 5. Also the line ℓ is perpendicular to the line given by the equation $5x - y = 15$. What is the length of the hypotenuse of T?

(A) $\sqrt{20}$ (B) $\sqrt{26}$ (C) $\sqrt{29}$ (D) $\sqrt{45}$ (E) $\sqrt{52}$

2. An army company is marching along at 3 mph down a straight road that is exactly 88 feet wide. The company is marching in a rectangular formation that is as wide as the road. Captain Allen likes to march at 5 mph, keeping just ahead of his company by zigzagging across the entire width of the road at the precise angle to stay exactly 2 yards in front of his troops at all times. If the company marches for 6 miles, approximately how many times will Captain Allen have gone from one side of the road to the other?

(A) 390 (B) 420 (C) 480 (D) 540 (E) 620

3. The figure below is built with eight line segments, each with either A or B as an endpoint. Triangles of various sizes are formed with parts of these segments as boundary.



How many triangular regions are there?

(A) 19 (B) 32 (C) 36 (D) 38 (E) 42

4. An $a \times b \times c,\ 2 \leqslant a \leqslant b \leqslant c$ rectangular block is built from abc unit cubes. From one corner you can see faces of three different sizes. Suppose you can see exactly 36 of the abc cubes. What is $a^2 + b^2 + c^2$?

(A) 41 (B) 50 (C) 55 (D) 60 (E) 65

5. Suppose a is a real number for which $a^2 + \dfrac{1}{a^2} = 14$. What is the largest possible value of $a^3 + \dfrac{1}{a^3}$?

(A) 48 (B) 52 (C) 56 (D) 60 (E) 64

6. Let $N = 7 + 77 + 707 + 7007 \cdots + 7 \cdot 10^{30} + 7$. When N is written in decimal (base 10) notation, what is the sum of the digits of N?

(A) 207 (B) 209 (C) 214 (D) 217 (E) 220

7. What is the largest power of 2 that divides the number $K = 75! - 71!$?

(A) 2^{62} (B) 2^{63} (C) 2^{65} (D) 2^{66} (E) 2^{67}

8. Let $p(x) = (x-7)(x^3 + 5x^2 + 1x - 11) + (x-9)(x^3 + 5x^2 + 1x - 11)$. What is the sum of the roots of $p(x) = 0$?

(A) -11 (B) -3 (C) 3 (D) 11 (E) 13

9. Suppose a, b, c are integers satisfying $1 \leqslant a < b < c$ and $a^2 + b^2 + c^2 = 14(a + b + c)$. What is the sum $a + b + c$?

(A) 38 (B) 39 (C) 40 (D) 41 (E) 42

10. What is the exact value of

$$\frac{\sqrt{13} - \sqrt{11}}{\sqrt{13} + \sqrt{11}} + \frac{\sqrt{13} + \sqrt{11}}{\sqrt{13} - \sqrt{11}}?$$

(A) 22 (B) $3(\sqrt{13} + \sqrt{11})$ (C) 24 (D) 25 (E) 26

11. What is the largest prime divisor of $2^{16} - 16$?

(A) 7 (B) 13 (C) 17 (D) 19 (E) 23

12. Suppose the roots of the equation $(x^2 - 2x + m)(x^2 - 2x + n) = 0$, where m, n are two real numbers, form an arithmetic sequence with the first term being $\dfrac{1}{4}$. Then $|m - n| =$

(A) 1 (B) $\dfrac{3}{4}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{8}$ (E) 2

13. If $f(x) = \dfrac{x^2+1}{x^2-1}$, then $f\left(\dfrac{1}{x}\right)$ is equal to:

(A) $f(x)$ (B) $-f(x)$ (C) $\dfrac{1}{-f(x)}$ (D) $\dfrac{-x^2-1}{1-x^2}$ (E) $\dfrac{1}{x}$

14. In the equation

$$\frac{X}{3} + \frac{Y}{4} = \frac{11}{12}$$

X, Y are natural numbers. Find $X+Y$.

(A) $X+Y=1$ (B) $X+Y=2$ (C) $X+Y=3$ (D) $X+Y=4$ (E) $X+Y=3$

15. Let ABC, DEF be two three digits numbers and different letters represent different digits, and neither the lead digits A and D is zero. Find the maximal difference

$$ABC - DEF$$

(A) 888 (B) 885 (C) 875 (D) 864 (E) 854

16. How many solutions does the equation $x^2 - [x^2] = \{x\}^2$ have on the interval $[1,3]$? Here $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts of a number.

(A) 8 (B) 7 (C) 6 (D) 5 (E) 4

17. How many real solutions does the equation $[x^3] + [x^2] + [x] = \{x\} - 1$ have? Here $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts of a number.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

18. Let $a_n = \dfrac{1}{\sqrt{n} + \sqrt{n+1}}$. Find the sum $a_1 + a_2 + \cdots + a_{99}$.

(A) 6 (B) 8 (C) 9 (D) 12 (E) 15

19. Evaluate exactly $\sqrt{5+2\sqrt{6}} + \sqrt{5-2\sqrt{6}}$.

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $2\sqrt{3}$ (D) $2\sqrt{5}$ (E) $\frac{3}{2}\sqrt{6}$

20. Let $f(x) = x^2+12x+30$. How many real solutions does the equation $f(f(f(f(f(x))))) = 0$ have?

(A) 0 (B) 1 (C) 2 (D) 3 (E) $\geqslant 4$

21. The sum of the first ten terms of a certain arithmetic sequence is 530 and the sum of the first twenty terms is 1860. What is the fifth term of the sequence?

(A) 48 (B) 49 (C) 50 (D) 51 (E) 52

22. The first three terms of a sequence are $2, 6, 60$ and each term afterwards is the product of the two previous terms, Fibonacci style, except we're multiplying. What is the number of decimal digits of the 6^{th} term of the sequence.

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

23. The sides of a right triangle are a, $2a + 2d$ and $2a + 3d$, with a and d both positive. The ratio of a to d is:

(A) $5 : 1$ (B) $27 : 2$ (C) $4 : 1$ (D) $1 : 5$ (E) $2 : 3$

24. Find the value of the product $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\ldots\left(1 - \frac{1}{225}\right)$.

(A) $\frac{4}{15}$ (B) $\frac{8}{15}$ (C) $\frac{16}{225}$ (D) $\frac{64}{225}$ (E) $\frac{128}{225}$

25. A rectangle consists of six squares. Find the side length of the biggest square if the side of the smallest square is 1.

(A) 5 (B) $\frac{49}{9}$ (C) 6 (D) $\frac{25}{4}$ (E) 7