Solutions

1. Christine deposited 3/5 of her savings into account A yielding 4% interest and she invested the rest into account B, yielding 3% interest. The interest earned in account A was 240 dollar higher than the interest earned in account B. How much money did she invest?

   (A) \$10,000 (B) \$20,000 (C) \$40,000 (D) \$120,000 (E) cannot be determined

   Answer: (B)

   Solution: Let X denote the capital. We have the following equation

$$\begin{aligned}\frac{3}{5}X(0.04) &= 240 + \frac{2}{5}X(0.03)\\ 0.12X &= 1200 + 0.06X\\ 0.06X &= 1200\\ X &= 20,000\end{aligned}$$

2. The pages of a book are numbered 1 through 2017. How many times does the digit 7 occur?

   (A) 602 (B) 202 (C) 402 (D) 405 (E) 302

   Answer: (A)

   Solution: The number 7 occurs in the ones place 202 times, in the tens place 200 times, and in the hundreds place 200 times for a total of 602 times.

3. Suppose that a and b are distinct real numbers such that each of them is one less than its reciprocal. What is their sum $a + b$?

   (A) -1 (B) 1 (C) 0 (D) $\frac{1}{2}$ (E) $-\frac{1}{2}$

   Answer: (A)

   Solution: if x is either of a, b then $x = \frac{1}{x} - 1$ and so $x^2 + x - 1 = 0$. But also, $(x - a)(x - b) = x^2 - (a + b)x + (ab) = x^2 + x - 1$, and $a + b = -1$.

4. Let $a \neq 1$ be a positive number. Calculate $I = \log_{\sqrt{a}} a$.

   (A) $I = \frac{1}{2}$ (B) $I = 0$. (C) $I = -2$. (D) $I = 2$. (E) $I = -\frac{1}{2}$

   Answer: (D)

   Solution: $I = \log_{\sqrt{a}} \sqrt{a}^2 = 2$.

5. What is the biggest whole number smaller than $\sqrt{2^{100} + 10^{10}}$?

   (A) 2^{50} (B) $2^{50} + 1$ (C) $2^{50} + 10^5 - 1$ (D) $2^{50} + 10^5$ (E) $2^{50} + 10^5 + 1$

   Answer: (A)

   Solution: First, notice that $10^3 = 1000 < 1024 = 2^{10}$. Raising both sides to the fifth power, we get $10^{15} < 2^{50}$. It follows that

$$\begin{aligned}(2^{50} + 1)^2 &= 2^{100} + 2 \cdot 2^{50} + 1 \\ &> 2^{100} + 2 \cdot 10^{15} + 1 > 2^{100} + 10^{10}.\end{aligned}$$

   Taking square roots, we see

$$\sqrt{2^{100} + 10^{10}} < 2^{50} + 1.$$

   As $2^{50} = \sqrt{2^{100}} < \sqrt{2^{100} + 10^{10}}$, the answer is (a).

6. Which of these numbers is the largest?

   (A) 10^{30000} (B) 3^{60000} (C) 2^{100000} (D) 1000^{1000} (E) $1000!$

   Answer: (C)

   Solution: Observe that

$$\begin{aligned}1000! &= 1000 \cdot 999 \cdot 998 \cdot \cdots \cdot 2 \cdot 1\\ &< 1000 \cdot 1000 \cdot 1000 \cdot \cdots \cdot 1000 \cdot 1000 \quad \text{(1000 times)}\\ &= 1000^{1000}\end{aligned}$$

   which shows that (E) < (D). Then observe that

$$1000^{1000} = (10^3)^{1000} = 10^{3\cdot 1000} = 10^{3000} < 10^{30000}$$

   which shows that (D) < (C). This means the answer is (A), (B), or (C). To compare (A) - (C), we can compare their ten thousandth roots. Because

$$3^6 < 10^3 < 2^{10}$$

   (which can be shown by a direct computation), we have

$$3^{60000} < 10^{30000} < 2^{100000}$$

   and the answer is (C).

7. Cities A and B are 160 miles apart. One car is traveling from city A toward city B, while a motorbike is simultaneously traveling the opposite direction, from city B toward city A. Two hours after the departure of both vehicles they meet somewhere in between, at point C. If motorbike's speed is 3/5 of the car's speed, find the distance of point C from city A.

   (A) 30 miles (B) 50 miles (C) 60 miles (D) 100 miles (E) 120 miles

   Answer: (D)

   Solution: Let x denote the car's speed, and y denote the motorbike's speed. Since the two vehicles travel toward each other, they will meet when the sum of their traveling distance is 160 miles. x and y denote speed, and distance is speed times time. Therefore, translating both quantities into distance, we have that $2x + 2y = 160$. Moreover, we have an additional information. We know that $y = (3/5)x$. Replacing this into our first equation, we get

$$\begin{aligned} 2x + 2y &= 160 \\ 2x + 2\frac{3}{5}x &= 160 \\ 16x = 800 &=> x = 50 \end{aligned}$$

   Since car's speed was 50 miles per hour and it was traveling for 2 hours, the distance that it covered is 100 miles. Therefore, the distance from city A to the meeting point is 100 miles.

8. The fifth term of an arithmetic sequence is 49 and the difference between successive terms of the sequence is 8. What is the sum of the first 20 terms of the sequence?

   (A) 177 (B) 500 (C) 1689 (D) 1860 (E) 2050

   Answer: (D)

   Solution: Let $a_k = b \cdot k + c$ be the k-th term in the sequence. The first sentence gives $b = 8$ and $a_5 = 8 \cdot 5 + c = 49$. Hence $c = 9$. Then $\sum_{k=1}^{20} a_k = 8 \cdot (20 \cdot 21/2) + 20 \cdot 9 = 1860$.

9. Jack and Lee walk around a circular track. It takes Jack and Lee respectively 6 and 10 minutes to finish each lap. They start at the same time, at the same point on the track, and walk in the same direction around the track. After how many minutes will they be at the same spot again (not necessarily at the starting point) for the first time after they start walking?

   (A) 15 (B) 16 (C) 30 (D) 8 (E) 60

   Answer: (A)

   Solution: Experimenting with the numbers in turn, dividing 6 into 15 and 10 into 15, we get answers one whole number apart, so they are together again at 15 minutes. Jack travels $t/6$ tracks per minute and Lee travels $t/10$. Jack must travel once more around than Lee, i.e. solve $t/6 = t/10 + 1$.

10. The polynomial $p(x) = x^4 + 2x^3 + ax^2 + 5x + b$ is divisible by $x^2 + x - 6$ for certain values of a and b. What is the sum of a and b?

    (A) -60 (B) 72 (C) -54 (D) 58 (E) -72

    Answer: (A)

    Solution: Because $x^2 + x - 6 = (x - 2)(x + 3)$, $p(x)$ has zeros at 2 and -3. Thus, $p(2) = 4a + b + 42 = 0$ and $p(-3) = 9a + b + 12 = 0$. Solve the equations, we have $a = 6$ and $b = -66$.

11. A man traveled from A to B at 40 miles an hour and then from B to A at 60 miles an hour. What was his average speed (in miles per hour) during the entire journey?

    (A) 46 (B) 48 (C) 50 (D) 52 (E) 54

    Answer: (B)

    Solution: Let d be the distance from A to B. The time from A to B is $d/40$. The time from B to A is $d/60$. The total time is therefore $(d/40) + (d/60) = d/24$. The average speed is distance divided by time, which is $(2d)/(d/24) = 48$.

12. How many real number solutions to $\sqrt{1+\sqrt{x}} = x - 1$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: (B)

Solution: First observe that the left-hand side is non-negative and in order for the right-hand side to be non-negative, we must have $x \geq 1$. Square both sides and subtract 1 to obtain $\sqrt{x} = x^2 - 2x$. The graph of the right-hand side is the parabola $y = x^2$ shifted to the right by 1 and down by 1. The graph of the left-hand side is a half of the parabola $x = y^2$ (with the axis along the y-axis) for which $y \geq 0$. By sketching the graphs one immediately comes to the conclusion that the graph have exactly one point of intersection, i.e. the answer is B.

For a more rigorous proof, let us introduce $z = \sqrt{x} \geq 0$. Then $z = z^4 - 2z^2$. The root $z = 0$ implies $x = 0$ which is not a solution of the original equation. Thus $z^3 - 2z - 1 = 0$. Here $z = -1$ is a root which is not related to the original equation. We divide by $z+1$ and obtain $z^2 - z - 1 = 0$ with the roots $z = (1 \pm \sqrt{5})/2$. The negative root is not relevant, and the positive root provides the root x of the original equation. The answer is B.

13. There are three digits A, B and C such that

$$AB + CB = BBA.$$

What is their sum?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Answer: (E)

Solution: E. $B = 1$, so $A = 2$ and $C = 9$.

14. Let n be a positive integer. Which of the numbers $\{1, 2, 3, 5, 6\}$ need not divide evenly into $n^3 + 5n$?

(A) 1 (B) 2 (C) 3 (D) 5 (E) 6

Answer: (D)

Solution: The correct answer is D. One can use induction to prove that $n^3 + 5n$ can be always divided by 6. Indeed, $(n+1)^3 + 5(n+1) = n^3 + 5n + 3(n+1)n + 6$. It is sufficient to see now that $3(n+1)n$ can be divided by 6.

15. Find digit values for the letters `T, W, O, E, I, G,` and `H` so that `2 (T W O)`^{2} `= E I G H T`. What is the value of `O`?

    (A) 4 (B) 5 (C) 6 (D) 8 (E) 9

    Answer: (E)

    Solution: First note that `T` is positive, even and less than 4. So `T` is 2. This means that `O` has a square whose units digit is 1 or 6. Why? We get 3 cases, `O=4,6,9`. Checking these one at a time we find that only `O=9` works. In this case, `T=2, W=0, O=9, E=8, I=7, G=3` and `H=6`.

16. Every integer has a nonary (Base 9) representation using only the digits $0, 1, \ldots, 8$. Let $f(x) = (2x+3)^2(x^4)$. What is the sum of the digits of the nonary representation of $f(9)$?

    (A) 8 (B) 9 (C) 10 (D) 11 (E) more than 11

    Answer: (B)

    Solution: B.

$$\begin{aligned} f(9) =& (21)^2(9^4) = 7^2 \cdot 9^5 = (45+4) \cdot 9^5 \\ =& 5 \cdot 9^6 + 4 \cdot 9^5 = 5400000_9 \ . \end{aligned}$$

17. Three integers a, b and c satisfy both $0 \le a \le b \le c$ and

$$abc + ab + ac + bc + a + b + c = 848.$$

    What is the smallest possible value of $a+b+c$?

    (A) 176 (B) 282 (C) 424 (D) 848 (E) There are no such triplets.

    Answer: (B)

    Solution: B. Add 1 to both sides and factor to get $(a+1)(b+1)(c+1) = 849$, which is a multiple of 3. Factoring $849 = (3)(283)$ and check that 283 is prime. So there are two solutions $a = 0, b = 2$ and $c = 282$, and $a = b = 0, c = 848$.

18. Find the minimum of $x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3$ as x_1, x_2, and x_3 range over all real numbers, independently.

(A) 0 (B) 2 (C) 1 (D) -1 (E) -2

Answer: (A)

Solution: **Answer:** (A)

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3 = \frac{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_1 - x_3)^2}{2}.$$

The right hand side is at least zero, and equality is attained when $x_1 = x_2 = x_3$.

19. The roots of the equation $x^3 - ax^2 + 5x - 1 = 0$ have the property that for each root r its multiplicative inverse $1/r$ is also a root. What is the value of a?

Answer: (C)

(A) 0 (B) -2 (C) 5 (D) 1 (E) 2

Solution: **Answer:** (C)

If all three roots are equal to 1, we get the polynomial $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ in which the coefficient of x is not 5. It is similarly easily checked that there is at least one root $r \neq \pm 1$. Therefore $1/r$ is also a root different from 1 and r and the third root must be 1. The polynomial we get is

$$(x - r) \cdot \left(x - \frac{1}{r}\right) \cdot (x - 1) = x^3 - \left(r + 1 + \frac{1}{r}\right) \cdot x^2 + \left(r + 1 + \frac{1}{r}\right) \cdot x - 1$$

Hence the coefficient of x^2 is the additive inverse of the coefficient of x.

20. What is the last digit of the number 7^{7^7}?

    Answer: (D)

    (A) 1 (B) 7 (C) 9 (D) 3 (E) 8

Solution: **Answer:** exploring the last digits of powers of 7: $7^0 = 1$, $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$; then cycle through $1, 7, 9, 3$ again. That is 7^{4k} has last digit 1, 7^{4k+1} has last digit 7, 7^{4k+2} has last digit 9, and 7^{4k+3} has last digit 3.

So, to determine the last digit of 7^{7^7}, we have to determine the remainder of $7^7/4$. That is, $7^7 = (2 \cdot 4 - 1)^7$ which equals $4 \cdot A - 1$ for some value A. And $4 \cdot A - 1 = 4 \cdot (A - 1) + 3$, so the remainder is 3. Which therefore means the last digit of 7^{7^7} is also 3.