March 7, 2016

1. The value of $\frac{(2\times10^{-3})(5\times10^5)}{(0.2\times10^2)(0.5\times10^2)}$ is

   (A) 10^{-9} (B) 10   (C) 10^2 (D) 10^{-1} (E) none of these

   Solution: (E)

   $$
   \frac{(2\times10^{-3})(5\times10^5)}{(0.2\times10^2)(0.5\times10^2)}
   $$
   $$
   = \frac{10\times10^2}{0.1\times10^4}
   $$
   $$
   = \frac{10^3}{10^3} = 1
   $$

2. The little tycoon Johnny says to his fellow capitalist Annie, "If I add 7 dollars to 3/5 of my funds, I'll have as much capital as you have." To which Annie replies, "So you have only 3 dollars more than me." How much money does Annie have?

   (A) 9   (B) 17   (C) 22   (D) 15   (E) 20

   Solution: (C)

   If we add 7 and then 3 dollars to 3/5 of Johnny's funds, we get his entire holdings. So 2/5 of his total equals 10 dollars. It follows that Johnny has 25 dollars and Annie has 22 dollars.

3. Four little girls, Katrina, Helen, Marie, and Naomi, sang in a concert. Each song was sung by three girls. Katrina sang 8 songs, which was more than the other girls. Helen sang 5 songs, which was fewer than the other girls. How many songs were sung in the concert?

   (A) 27   (B) 9   (C) 10   (D) 6   (E) 15

   Solution: (B)

   9 songs. Suppose we gave a candy to each girl who sang a song. Since every song was sung by three girls, the number of all candies must be divisible by 3. The number of candies given might be: 8+7+7+5 = 27, or 8+7+6+5 = 26, or 8+6+6+5 = 25. But only the number 27 is divisible by 3, and, therefore, the number of songs must be 27/3 = 9.

4. A state program allows people to collect empty milk bottles and exchange them for bottles full of milk. Four empty bottles may be exchanged for one full bottle. How many bottles of milk can a family drink if it has collected 24 empty bottles?

(A) 6   (B) 7   (C) 8   (D) 9   (E) 12

Solution: (C)

The family can drink $6 + 1 = 7$ bottles of milk, and it will have three empty bottles left. Then the family can borrow one empty bottle, exchange the four empty bottles for one bottle of milk, drink it, and return the bottle it borrowed. Thus, the family can drink eight bottles of milk.

5. Nick left Nicktown at 10:18 a.m. and arrived at Georgetown at 1:30 p.m., walking at a constant speed. On the same day, George left Georgetown at 9:00 a.m. and arrived at Nicktown at 11:40 a.m., walking at a constant speed along the same road. The road crosses a wide river. Nick and George arrived at the bridge simultaneously, each from his side of the river. Nick left the bridge one minute later than George. When did they arrive at the bridge?

(A) $10:45$ (B) $11:20$ (C) $11:00$ (D) $2:10$ (E) $1:30$

Solution: (C)

It took Nick 3 hours 12 minutes - that is, $16/5$ hours - to reach Georgetown, and it took George 2 hours 40 minutes - that is, $8/3$ hours - to reach Nicktown. Denoting the distance between the towns by L miles, we find that Nick was walking at a speed of $5L/16$ mph and George's speed was $3L/8$ mph. We can determine the length of the bridge l, since we know that George crossed it one minute faster than Nick: $16l/5L - 8l/3L = 1/60$. This yields $l = L/32$. Let t be the moment the boys reached the bridge. At this moment, the total distance walked by both boys was $L - L/32 = 31L/32$. On the other hand, this equals the sum of the distances walked by each of them - that is, $(5L/16)[t - (10 + 3/10)] + (3L/8)(t - 9) = (L/16)(11t - 211/2)$. Setting these expressions equal to each other, we obtain $(L/16)(11t - 211/2) = 31L/32$, which gives us $t = 11$ o'clock.

6. A man is filling two tanks with water using two hoses. When the smaller tank is half full, he switches hoses. He keeps filling the tanks, and they both fill up completely at the same moment. The smaller tank is filled first from the less powerful hose. The first hose delivers water at the rate of 2.9 liters per minute, the second at a rate of 8.7 liters per minute. What is the volume of the larger tank if the volume of the smaller tank is 12.6 liters?

(A) 19.3   (B) 18.2   (C) 25.2   (D) 30   (E) 21

Solution: (E)

It will take 6.3/2.9 minutes to fill the first half of the small container, during which the larger container receives (6.3/2.9)(8.7) liters of water. Then it will take 6.3/8.7 minutes to fill the second half of the small container, and the same amount to top off the larger one (because the jobs are completed at the same time). So the larger container, during this time, receives (6.3/8.7)(2.9) liters of water. Since this fills the larger container, it must contain (6.3/2.9)(8.7) + (6.3/8.7)(2.9) = 21 liters.

7. Let a and b be the two solutions to the equation

$$2x^2 - 3x - 3 = 0.$$

Find the value $\frac{1}{a} + \frac{1}{b}$.

(A) 2   (B) 1   (C) 0   (D) −1   (E) 5

Solution: (D)

We have $2(x - a)(x - b) = 2x^2 - 2x(a + b) + 2ab$. Hence $2(a + b) = 3, 2ab = -3$. Then $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = -1$.

8. What is the sum of the digits of all numbers from 1 to 1000?

(A) 13501   (B) 13601   (C) 13701   (D) 13801   (E) 13901

Solution: (A)

Let us add zero to the set of the numbers and divide the set into 500 pairs $(0, 999)$, $(1, 998)$, ..., $(499, 500)$. The sum of the digits in each pair is 27. In addition, we need to take into account 1000, so that the total sum of all digits will be $27 \cdot 500 + 1 = 13501$.

9. How many integers n satisfy $|n^3 - 222| < 666$?

(A) 11   (B) 15   (C) 17   (D) 19   (E) 20

Solution: (C)

The given equation is equivalent to $-666 < n^3 - 222 < 666$ which in turn is equivalent to $-444 < n^3 < 888$, which has the integral solutions $-7, -6, ..., 0, 1, 2, ..., 9$ of which there are 17.

10. Which of the five fractions is smallest?

(A) $\frac{250388749}{250388751}$ (B) $\frac{250388748}{250388750}$ (C) $\frac{250388747}{250388749}$ (D) $\frac{250388750}{250388751}$

(E) $\frac{250388751}{250388752}$

Solution: (C)

Follows from the inequality $\frac{x}{y} < \frac{x+1}{y+1}$ for $0 < x < y$.

11. A total of n cards numbered 1 through n is divided into two stacks. What is the minimum value of n such that at least one stack will include a pair of cards whose numbers add up to an exact square?

    (A) 36   (B) 15   (C) 52   (D) 48   (E) 12

    Solution: (B)

    The minimum value of n is 15. First, we prove that for 15 cards, the desired pair can be found. Suppose the contrary. Then the cards numbered 1 and 15 must be in different stacks, as must cards 1 and 3. Thus cards 3 and 15 are in the same stack. Therefore, cards $6 = 93$ and $10 = 2515$ are in the other stack, which contradicts the assumption, since $6 + 10 = 16$. Now we show that 14 cards can be distributed between the two stacks such that the sum of the numbers of any two cards of the same stack is not an exact square. Here is an example: $1, 2, 4, 6, 9, 11, 13$ (the first stack) and $3, 5, 7, 8, 10, 12, 14$ (the second stack). For any number of cards less than 14, the cards can be distributed between the two stacks in a similar way (with the desired condition holding true).

12. A bug crawls along the edges of a cube. Each time it gets to a vertex, it chooses one of the three edges leaving that vertex. How many of the $3^4 = 81$ paths of length 4 lead back to the original vertex?

    (A) 81   (B) 4   (C) 8   (D) 21   (E) 37

    Solution: (D)

    Each term of the polynomial $(x + y + z)^4$ corresponds to a set of paths. The sum of all the coefficients of the terms $x^i y^j z^k$, where all coefficients i, j, k are even and the sum $i + j + k = 4$, is $1 + 1 + 1 + 6 + 6 + 6 = 21$.

13. The outside of an $a \times b \times c$ block of unit cubes is painted, where $a < b < c$. Exactly two-thirds of the abc cubes have some paint. Which of the following could be (a, b, c)?
    (A) (5,7,9)   (B) (6,8,10)   (C) (7,9,11)   (D) (6,10,12)   (E) (7,10,12)

    Solution: (A)

    The number of cubes without paint is $a \cdot b \cdot c/3$, and this must be the same as $(a-2)(b-2)(c-2)$, which in the first case reduces to $\frac{1}{3} = \frac{3}{5} \times \frac{5}{7} \times \frac{7}{9}$.

14. What is the remainder when the product $N = 1008 \cdot 1009 \cdot 1010$ is divided by 77?

    (A) 18   (B) 27   (C) 42   (D) 63   (E) 65

    Solution: (C)

Since $1008 = 1001 + 7$ is a multiple of 7, we need only compute the residue of N modulo 11. But $1008 \equiv 7 \pmod{11}$, $1009 \equiv 8 \pmod{11}$, and $1010 \equiv 9 \pmod{11}$, so the product is congruent to 9 modulo 11. Thus the remainder must be $42 = 3 \times 11 + 9$.

15. Find the sum
$$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + 2016! \cdot 2016$$

    (A) $2017! \cdot 2016$ (B) $2017! - 1$ (C) $2017! - 2016$ (D) $2017! + 1$ (E) $2017! + 2016$

    Solution: (B)

    Note that $(n+1)! = n! \cdot (n+1) = n! \cdot n + n!$, and therefore $n! \cdot n = (n+1)! - n!$ Then we can rewrite our expression as a telescopic series:

    $$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + 2016! \cdot 2016$$

    $$= (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + (2017! - 2016!) = 2017! - 1$$

16. If a, b, c, and d are nonzero real numbers, $a/b = c/d$, and $a/d = b/c$, then which one of the following must be true?

    (A) $a = \pm b$ (B) $a = \pm c$ (C) $a = \pm d$ (D) $b = \pm c$ (E) none of A, B, C or D

    Solution: (A) From the second equation, we have $\frac{a}{b} = \frac{d}{c}$. But the first equation asserts that $\frac{a}{b} = \frac{c}{d}$. Thus $\frac{a}{b} = \pm 1$. To see that none of the other choices can be correct, let $a = 1, b = -1, c = 2$, and $d = -2$.

17. Suppose $ab < 0$. Which of the following points could not satisfy $y = ax + b$?

    (A) $(0, 1)$ (B) $(1, 0)$ (C) $(-1, 0)$ (D) $(0, -1)$ (E) $(1, 1)$

    Solution: (C)

    The lines $y = -x + 1, y = x - 1$, and $y = -x + 2$ contain the points $(0, 1), (1, 0), (0, 1)$, and $(1, 1)$, but $(-1, 0)$ cannot belong to such a line. If a line with positive slope contains $(-1, 0)$, that line must also have a positive y-intercept, and if a line with negative slope contains $(-1, 0)$, that line must also have a negative y-intercept.

18. Let x and y be two real numbers satisfying $x + y = 6$ and $xy = 7$. Find the value of $x^3 + y^3$.

    (A) 55   (B) 62   (C) 78   (D) 90   (E) 216

    Solution: (D)

    Expand $(x+y)^3$ to yield $216 = 6^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x+y) = x^3 + y^3 + 3 \times 7 \times 6 = x^3 + y^3 + 126$. So $x^3 + y^3 = 216 - 126 = 90$.

19. Suppose a and b are positive integers neither of which is a multiple of 3. Then the remainder when $a^2 + b^2$ is divided by 3

    (A) must be 0   (B) must be 1   (C) must be 2   (D) may be 1 or 2 but not 0
    (E) may be 0, 1 or 2

    Solution: (C)

    Since $(3n+1)^2 = 9n^2 + 6n + 1$ and $(3n+2)^2 = 9n^2 + 12n + 4$ are both one bigger than a multiple of 3, the sum $a^2 + b^2$ must be two times bigger than a multiple 3.

20. A 3 by 3 by 3 wooden cube is painted on all 6 faces and then cut into 27 unit cubes. One unit cube is randomly selected and rolled. What is the probability that exactly two of the five visible faces are painted?

    (A) $\frac{1}{27}$ (B) $\frac{2}{27}$ (C) $\frac{12}{27}$ (D) $\frac{15}{27}$ (E) $\frac{5}{81}$

    Solution: (C)

    Such cube consists of 27 small cubes with 8 three sides painted, 12 with two sides painted, and 6 with one side painted, and one unpainted. Using total probability formula we have P(exactly two painted) $= P$(two-sided selected)$\times P$(falls on unpainted side)$+P$(three-sided selected)$\times P$(falls on painted side)$= (12/27)(4/6) + (8/27)(3/6) = 12/27$.

21. The solution set of $8x^6 + 7x^4 + 6x^2 + 5 < 0$ is

    (A) $\varnothing$ (B) $\{x | x < 0\}$ (C) $\{0, 1\}$ (D) $\{x | x > 0\}$ (E) none of these

    Solution: (A)

    Since $x^6 \geq 0$, $x^4 \geq 0$ and $x^2 \geq 0$, $8x^6 + 7x^4 + 6x^2 + 5 \geq 5$. Therefore, the solution set of $8x^6 + 7x^4 + 6x^2 + 5 < 0$ is $\varnothing$.

22. If f is a function such that $f(3) = 2$, $f(4) = 2$ and $f(n + 4) = f(n + 3) \cdot f(n + 2)$ for all the integers $n \geq 0$, what is the value of $f(6)$?

    (A) 4   (B) 5   (C) 6   (D) 8   (E) it cannot be determined from the information given

    Solution: (D)

    $f(6) = f(2 + 4) = f(5) \times f(4) = (f(4) \times f(3)) \times f(4) = 2 \cdot 2 \cdot 2 = 8$.

23. From a group of three female students and two male students, a three student committee is selected. If the selection is random, what is the probability that exactly 2 females and 1 male are selected?

    (A) 0.3   (B) 0.4   (C) 0.5   (D) 0.6   (E) 0.7

Solution: (D)

There are $\binom{5}{3} = 10$ ways to select the committee. One of these has three females and three have two males. The other all have two females and one males. Thus the probability is 0.6.

24. When a missile is fired from a ship, the probability it is intercepted is $\frac{1}{3}$. The probability that the missile hits the target, given that it is not intercepted, is $\frac{3}{4}$. If three missiles are fired independently, what is the probability that all three hit their targets?

   (A) $\frac{1}{12}$ (B) $\frac{1}{8}$ (C) $\frac{9}{64}$ (D) $\frac{3}{8}$ (E) $\frac{3}{4}$

   Solution: (B)

   The probability that a missile is not intercepted is 2/3. The probability that any one missile hits its target is $P(\text{hits})= (3/4) \times (2/3) = 1/2$. The three missiles are fired independently, so the desired probability is $1/2 \times 1/2 \times 1/2 = 1/8$.

25. A cubic polynomial $p(x)$ with leading coefficient 1 has three zeros, $x = 1, x = -1$, and $x = 3$. What is the value $p(2)$?

   (A) -3 (B) -1 (C) 1 (D) 2 (E) 3

   Solution: (A)

   The polynomial is $p(x) = a(x - 1)(x + 1)(x - 3)$, and since the leading coefficient of p is 1, it follows that $a = 1$. Thus $p(2) = 1 \times 3 \times (-1) = -3$.