UNC Charlotte 2010 Comprehensive

March 8, 2010

1. A cubic equation $x^3 - 4x^2 - 11x + a = 0$ has three roots, x_1, x_2, x_3. If $x_1 = x_2 + x_3$, what is a?

   (A) 24 **(B)** 28 **(C)** 30 **(D)** 32 **(E)** 36

2. For which value of a is the polynomial $P(x) = x^{1000} + ax + 9$ divisible by $x + 1$?

   (A) *none* **(B)** 10 **(C)** -10 **(D)** 9 **(E)** -9

3. Suppose you visit Mars and meet some aliens who teach you their system of counting. You notice that they use a true place value system that is similar to ours, but the Martians use base 6 instead of base 10. Table 1 shows how to translate their characters into our digits.

#	&	<	@	/	*
0	1	2	3	4	5

Table 1: Martian characters

   Convert the number 54 (base 10) to its Martian equivalent.

   (A) &@ **(B)** &@# **(C)** @& **(D)** #@& **(E)** */

4. Let $x = \dfrac{1}{2}^{\left(\frac{1}{2}^{-\frac{1}{2}}\right)}$. To which of the following intervals does x belong?

   (A) $(0, 1/8]$ **(B)** $(1/8, 1/4]$ **(C)** $(1/4, 1/2]$ **(D)** $(1/2, 1]$ **(E)** $(1, \infty)$

5. A room is shaped like a cube with sides of length 4 meters. Suppose that A and B denote two corners of the room that are farthest from each other. A caterpillar is crawling from A to B along the walls. What is the shortest possible length of the trip?

   (A) $4\sqrt{2}$ meters **(B)** $4\sqrt{3}$ meters **(C)** 8 meters

   (D) $4\sqrt{5}$ meters **(E)** $4\sqrt{6}$ meters

6. A plane passes through the center of a cube and is perpendicular to one of the cube's diagonals. How many edges of the cube does the plane intersect? (An edge is a line connecting adjacent corners.)

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

7. Four numbers are written in a row. The average of the first two numbers is 5. The average of the middle two numbers is 4 and the average of the last two numbers is 10. What is the average of the first and last numbers?

   (A) 9 **(B)** 10 **(C)** 10.5 **(D)** 11 **(E)** 11.5

8. The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Squares containing numbers do not contain mines. Each square that does not have a number either has a single mine or nothing at all. How many mines are there?

   (A) 19 **(B)** 20 **(C)** 21 **(D)** 22 **(E)** 23

	1		1		2		2		1
1		2		3		2		3	
	3		2		3		2		2
1		3		2		1		3	
	3		2		1		1		2
2		4		1		1		2	
	3		3		2		3		1
1		2		2		3		2	

9. John and Bill toss a biased coin that has a 60% chance of coming up heads and a 40% chance of coming up tails. They flip the coin until either two heads or two tails in a row are observed. Bill is a winner if two heads in a row are observed first. Which of the following numbers is closest to the probability that Bill will win?

   (A) 61/95 **(B)** 63/95 **(C)** 64/95 **(D)** 67/95 **(E)** 69/95

10. How many times in a 24 hour period do the hour and minute hands of a clock form a right angle?

    (A) 48 **(B)** 44 **(C)** 34 **(D)** 24 **(E)** None of these.

11. A pyramid has a 10 ft. by 10 ft. square base. The height of the pyramid is also 10 ft. If we want to slice the pyramid into two pieces with equal volumes by cutting it with a plane parallel to the base, how far above the base should we make the cut?

    (A) 5 ft. **(B)** $5 - \sqrt[3]{4}$ ft. **(C)** $10 - 5\sqrt[3]{4}$ ft.

    (D) $\frac{10}{3}$ ft. **(E)** None of these.

12. How many pairs (x, y) of positive integers satisfy $2x + 7y = 1000$?

    (A) 70 **(B)** 71 **(C)** 72 **(D)** 73 **(E)** 74

13. Let A be the point $(7, 4)$ and D be the point $(5, 3)$. What is the length of the shortest path $ABCD$, where B is a point $(x, 2)$ and C is a point $(x, 0)$? This path consists of three connected segments, with the middle one vertical.

    (A) $2 + \sqrt{29}$ **(B)** $\sqrt{31}$ **(C)** $2 + \sqrt{31}$ **(D)** $2 + \sqrt{33}$ **(E)** $\sqrt{41}$

14. Suppose $f(0) = 3$ and $f(n) = f(n-1) + 2$. Let $T = f(f(f(f(5))))$. What is the sum of the digits of T?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

15. Four sets A, B, C, and D each have 500 elements. The intersection of any two of the sets has 115 elements. The intersection of any three of the sets has 52 elements. The intersection of all four sets has 30 elements. How many elements are there in the union of the four sets?

    (A) 1465 **(B)** 1472 **(C)** 1482 **(D)** 1488 **(E)** 1512

16. How many three digit numbers can be written as a sum of a three digit number and its (one-, two-, or three-digit) reversal?

    (A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

17. If (x, y, z) satisfy the three equations below, what is $x + y + z$?

$$\left.\begin{array}{rcl} 2x - y + z &=& 7 \\ x + 2y + 3z &=& 1 \\ -x + y + 5z &=& 6 \end{array}\right\}$$

    (A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

18. How many positive integer divisors does $N = 250 \cdot 88$ have?

    (A) 24 **(B)** 28 **(C)** 30 **(D)** 32 **(E)** 40

19. A $4\times4\times4$ wooden cube is painted on *five* of its faces and is then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that the face showing is painted? Express your answer as a fraction.

    (A) 5/24 **(B)** 1/4 **(C)** 7/24 **(D)** 1/3 **(E)** 1/2

20. How many integer triples (x, y, z) satisfy both

    - x, y, and z are positive integers less than 30 and
    - $xy^2z^3 = 10,000$.

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

21. What is the sum of the three positive integers a, b, and c that satisfy

$$a + \frac{1}{b + \frac{1}{c}} = 5.4?$$

    (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

22. Use each of the digits $2, 3, 4, 6, 7, 8$ exactly once to construct two three-digit numbers M and N so that $M - N$ is positive and is as small as possible. Compute $M - N$.

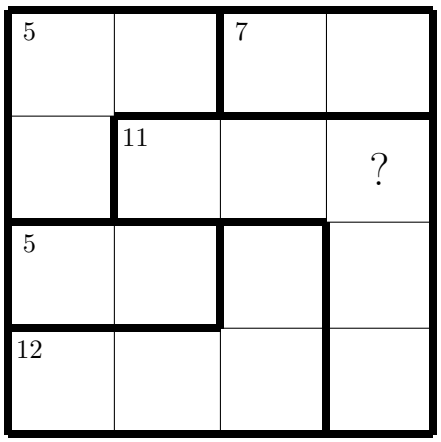
    (A) 19 **(B)** 29 **(C)** 39 **(D)** 49 **(E)** 59

23. How many integers x with $1 \leq x \leq 100$ satisfy the equation $x^2 + x^3 = y^2$ for some integer y?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

24. Consider the 4×4 Kenken® puzzle below. The solution uses the numbers 1 to 4 exactly once in each row and each column. The sum of the digits in each *cage* is the number given in the upper left corner of one of the squares. What digit goes in the square with the '?' ?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** The given puzzle has no solution.



25. $ABCD$ is an isosceles trapezoid with $\overline{AB}$ parallel to $\overline{DC}$, $AC = DC$, and $AD = BC$. If the height h of the trapezoid is equal to AB, find the ratio $AB : DC$.

    (A) 2 : 3 **(B)** 3 : 5 **(C)** 4 : 5 **(D)** 5 : 7 **(E)** 5 : 9