UNC Charlotte 2007 Algebra

March 5, 2007

1. We randomly select 4 prime numbers without replacement from the first 10 prime numbers. What is the probability that the sum of the four selected numbers is odd?

   (A) 0.21 **(B)** 0.30 **(C)** 0.36 **(D)** 0.40 **(E)** 0.50

   Solution: D. The event happens precisely when the number 2 is one of the primes selected. This occurs with probability $\binom{9}{3} \div \binom{10}{4} = \frac{9\cdot8\cdot7}{3\cdot2\cdot1} \cdot \frac{4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7} = 0.40$.

2. What is the area (in square units) of the region in the first quadrant defined by $18 \le x + y \le 20$?

   (A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** 44

   Solution: B. In the first quadrant, the inequalities $x+y \le 20$ and $x+y \le 18$ define similar right triangles with areas $20^2/2$ and $18^2/2$. The area of the region in question is therefore $(20^2 - 18^2) \div 2 = 38$ square units.

   Alternatively, we can also find the area as the area of a trapezoid. The parallel sides have lengths $20\sqrt{2}$ and $18\sqrt{2}$ and the height of the trapezoid is $\sqrt{2}$. Thus the area is $(38\sqrt{2}/2) \cdot \sqrt{2} = 38$.

3. How many four-digit numbers between 6000 and 7000 are there for which the thousands digits equal the sum of the other three digits?

   (A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

   Solution: E. The question is equivalent to 'how many three digit numbers $\underline{abc}$ (0 allowed as the hundreds digit) satisfy $a+b+c=6$. When $a=6$, there is only one way to do this: $b=c=0$. When $a=5$, there are two ways: either $b=1$ and $c=0$ or $c=1$ and $b=0$. When $a=4$, there are three ways: $(b,c)=(0,2),(1,1)$, or $(2,0)$. Continuing in this way, we find that there are $1+2+3+\ldots+7=28$ ways to build the number.

   Alternatively, there are three ways to choose three different digits that sum to 6: $1+2+3$, $0+2+4$ and $0+1+5$. Since the order counts, this gives 18 ways. There are three ways to choose two digits the same and a different third digit: $1+1+4$, $0+3+3$ and $0+0+6$. Since two of the digits are the same, there are 9 ways of ordering these. Finally, there is one sum, $2+2+2$, where all three digits are the same. So the total number of ways is $18+9+1=28$.

4. How many positive two-digit integers have an odd number of positive divisors?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution: D. Since each divisor d of a number D can be paired with a divisor D/d, only the perfect squares can have an odd number of divisors. There are 6 perfect squares between 10 and 99.

5. If x is positive, what is the least value of $x + \frac{9}{x}$?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

   Solution: E. Note: $x+\frac{9}{x} = \frac{1}{x}(x^2+9) = \frac{1}{x}(x^2-6x+9)+6 = \frac{1}{x}(x-3)^2+6 \geq 6$.

6. The area of an annular region bounded by two concentric circles is 5π square centimeters. The difference between the radii of the circles is one centimeter. What is the radius of the smaller circle, in centimeters?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

   Solution: B. If r is the radius of the smaller circle, the area **of the annular region** is $(r+1)^2\pi - r^2\pi = (2r+1)\pi$. The solution of the equation $2r+1 = 5$ is $r = 2$.

   Alternatively, let s be the radius of the larger circle Then $5\pi = s^2\pi - r^2\pi = (s-r)(s+r)\pi$. Since $s - r = 1$, $s + r = 5$. So $s = 3$ and $r = 2$.

7. If we divide 344 by d the remainder is 3, and if we divide 715 by d the remainder is 2. Which of the following is true about d?

   (A) $10 \leq d \leq 19$ **(B)** $20 \leq d \leq 29$ **(C)** $30 \leq d \leq 39$

   (D) $40 \leq d \leq 49$ **(E)** $50 \leq d \leq 59$

   Solution: C. The number d divides $341 = 11 \cdot 31$ and $713 = 23 \cdot 31$. The only common divisors are 1 and 31. Since we get nonzero remainders, $d = 31$.

8. The sum $a+b$, the product $a \cdot b$ and the difference of squares $a^2 - b^2$ of two positive numbers a and b is the same nonzero number. What is b?

   (A) 1 **(B)** $\frac{1+\sqrt{5}}{2}$ **(C)** $\sqrt{3}$ **(D)** $\frac{7-\sqrt{5}}{2}$ **(E)** 8

   Solution: B. Since $a+b = a^2 - b^2 = (a+b)\cdot(a-b)$, dividing both sides by $a+b \neq 0$ yields $1 = a-b$, so $a = b+1$. Substituting this into $a+b = a\cdot b$ we get $2b+1 = (b+1)\cdot b$. The only positive solution of this quadratic equation is

   $$b = \frac{1+\sqrt{5}}{2}.$$

9. It takes Amy and Bill 15 hours to paint a house, it takes Bill and Chandra 20 hours, and it takes Chandra and Amy 30 hours. How long will it take if all three work together?

   (A) 9 hours and 40 minutes **(B)** 10 hours **(C)** 12 hours

   (D) 13 hours and 20 min **(E)** 14 hours

   Solution: D. If Amy, Bill and Chandra can paint a whole house at a rate of $1/a, 1/b$, and $1/c$ of the house per hour, respectively, then $1/a + 1/b = 1/15$, $1/b + 1/c = 1/20$, and $1/c + 1/a = 1/30$. When all three work together, they can paint at a rate of $(1/a+1/b+1/c) = 1/2(1/15+1/20+1/30) = 3/40$. So it will take 40/3 hours, that is, 13 hours 20 min.

   Alternatively, let A be the rate Amy paints, B be the rate bill paints and C be the rate Chandra paints. Then $15A + 15B = 1$ house, $20B + 20C = 1$ house and $30A + 30C = 1$ house. So $A+B = 1/15$, $B+C = 1/20$ and $A+C = 1/30$. Thus $2(A+B+C) = (A+B)+(B+C)+(A+C) = 1/15+1/20+1/30 = 3/20$. Therefore $(40/3)(A+B+C) = 1$ house. So the time required is 40/3 hours or 13 hours and 20 minutes.

10. Maya deposited 1000 dollars at 6% interest compounded annually. What is the number of dollars in the account after four years?

    (A) \$1258.47 **(B)** \$1260.18 **(C)** \$1262.48

    (D) \$1263.76 **(E)** \$1264.87

    Solution: C. The amount in the account is given by $A = P(1+r/n)^{nt}$, where P is the principal, r is the annual rate, n is the number of times per year that interest is compounded, and t is the time in years. Therefore, the amount in the account after four years is $1000(1+.06/1)^4 \approx 1262.48$.

11. A peddler is taking eggs to the market to sell. The eggs are in a cart that holds up to 500 eggs. If the eggs are removed from the cart either $2, 3, 4, 5$, or 6 at a time, one egg is always left over. If the eggs are removed 7 at a time, no eggs are left over. Let n denote the number of eggs in the cart. Which of the following is true about n?

    (A) $n \in [1, 100]$ **(B)** $n \in [101, 200]$ **(C)** $n \in [201, 300]$

    (D) $n \in [301, 400]$ **(E)** $n \in [401, 500]$

    Solution: D. The first few facts can be interpreted as saying $n = 4i + 1, n = 5j + 1$ and $n = 6k + 1$, where i, j and k are integers. This means that $n - 1$ must be a multiple of 4, 5, and 6. Thus $n - 1 = 60m$ for some integer m. But n is a multiple of 7 also. Dividing each of $61, 121, 181, 241, 301$ by 7, we finally find that 301 is a multiple of 7.

12. An athlete covers three consecutive miles by swimming the first, running the second and cycling the third. He runs twice as fast as he swims and cycles one and a half times as fast as he runs. He takes ten minutes longer than he would if he cycled the whole three miles. How many minutes does he take?

    (A) 16 **(B)** 22 **(C)** 30 **(D)** 46 **(E)** 70

    Solution: B. Let S denote the time in hours required to swim a mile. Then the time required to run a mile is $\frac{S}{2}$, and the time needed to cycle a mile is $\frac{S}{3}$. It follows that $S + \frac{1}{2}S + \frac{1}{3}S - \frac{1}{6} = 3(\frac{1}{3}S)$, so $S = \frac{1}{5}$, and $S + \frac{1}{2}S + \frac{1}{3}S = \frac{11}{6}\frac{1}{5} = \frac{11}{30}$ hours, which is 22 minutes.

    Alternatively, let s be the swimming speed, r the running speed and c the cycling speed in miles per minute. Then his total time for the three miles is $t = (1/s) + (1/r) + (1/c) = 10 + (3/c)$. Also, $r = 2s$ and $c = 1.5r$. So $c = 3s$, $1/s = 3/c$ and $1/r = 3/2c$. Subbing in yields $5/2c = 10$ and $1/c = 4$. So the total time is $10 + 3(4) = 22$ minutes.

13. How many 5-digit numbers can be built using the digits $1, 2$, and 3 if each digit must be used at least once?

    (A) 60 **(B)** 90 **(C)** 120 **(D)** 150 **(E)** 243

    Solution: D. There are two patterns $1 - 1 - 3$, where we use one digit three times, and $2 - 2 - 1$, where we use two digits twice. There are $\binom{3}{1}\binom{5}{2} \cdot 2 = 3 \cdot 10 \cdot 2 = 60$ in the first group, and $+\binom{3}{1} \cdot 5 \cdot \binom{4}{2} = 3 \cdot 5 \cdot 6 = 90$ in the second group for a total of $60 + 90 = 150$.

14. What is the fewest crickets that must hop to new locations so that each row and each column has three crickets? Crickets can jump from any square to any other square.

♦	♦	♦	♦	
♦	♦	♦		♦
♦	♦			
♦	♦		♦	
		♦	♦	

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: C. Notice that two rows have four crickets, so at least two crickets must move. The pair of crickets at $(1, 5)$ and $(2, 4)$ on the main diagonal can be moved to $(5, 3)$ and $(5, 1)$ as shown.

	♦	♦	♦	
♦		♦		♦
♦	♦			♦
♦	♦		♦	
		♦	♦	♦

15. How many real number solutions does the equation $5\sqrt{x} = 6 - x$ have?

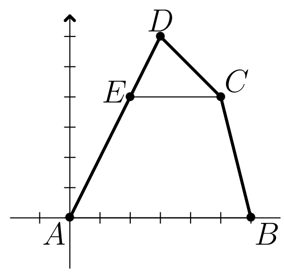
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** None of these

Solution: B. Squaring both sides and simplifying gives $x^2 - 37x + 36 = (x - 36(x - 1) = 0$. So there are two solutions, $x = 36$ and $x = 1$. However, checking for extraneous roots, only $x = 1$ satisfies the original equation.

16. A quadrilateral $ABCD$ has vertices with coordinates $A\,(0,0)$, $B\,(6,0)$, $C\,(5,4)$, $D\,(3,6)$. What is its area?

    (A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

    Solution: D. The point $E = (2,4)$ lies on AD. The line segment from $(2,4)$ to $(5,4)$ divides the quadrilateral into a trapezoid with area $\left(\frac{6+3}{2}\right)4$ and a triangle with area $\frac{1}{2}(3)(2)$. The total area is 21.

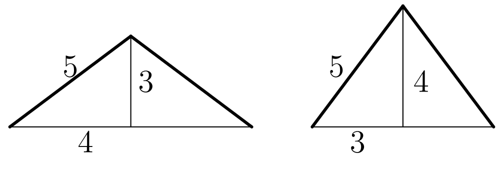


17. Let A be the area of a triangle with sides $5, 5$, and 8, and let B denote the area of a triangle with sides $5, 5$, and 6. Which of the following statements is true?

    (A) $A < B < 12$ **(B)** $B < A < 12$ **(C)** $A = B$

    (D) $12 < A < B$ **(E)** $12 < B < A$

    Solution: C. Each triangle has area 12. To see this, construct for each triangle the altitude to the even length side and use the Pythagorean Theorem.



18. In a course "Leadership in Mathematics" there are several tests. Each test is worth 100 points. After the last test John realized that if he had received 97 points for the last test, his average score for the course would have been a 90, and that if he had made a 73, his average score would have been an 87. How many tests are there in the course?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    Solution: E. Denote by N the number of tests in the course and by S the sum of John's scores for all but the last test. Then we can write the following system of equations:

$$\begin{aligned} \frac{S+97}{N} &= 90 \\ \frac{S+73}{N} &= 87 \end{aligned}$$

    Solving this system for N we obtain $N = 8$.

    Alternatively, Since the difference in the averages is 3 and the difference in the last tests is 24, $3 = 24/N$ where n is the total number of exams. So $N = 8$.

19. There are four consecutive integers such that the sum of the cubes of the first three numbers equals the cube of the fourth number. Find the sum of the four numbers.

    (A) 12 **(B)** 16 **(C)** 18 **(D)** 22 **(E)** 24

    Solution: C. The sum of four consecutive numbers $a+a+1+a+2+a+3 = 4a+6$, which is 6 bigger, hence 2 bigger, than a multiple of 4. Among the options, only 18 and 22 satisfy this requirement. Thus we need only check to see if $3^3+4^3+5^3 = 6^3$ or $4^3+5^3+6^3 = 7^3$. The former holds, so the answer is $3+4+5+6 = 18$.

    Alternatively, denote the smallest by $n-1$. Then $(n-1)^3+n^3+(n+1)^3 = (n+2)^3$. Expand and combine terms to have first $n^3-3n^2+3n-1+n^3+n^3+3n^2+3n+1 = n^3+6n^2+12n+8$, then $2n^3-6n^2-6n-8 = 2(n-4)(n^2+n+1)$. So $n = 4$ and the numbers are $3, 4, 5$ and 6 with sum 18.

20. Which of following statements is true about the equation $|x^2-2x-3| = x+2$?

    (A) There are no solutions. **(B)** There is only one solution.

    (C) There are exactly two solutions. **(D)** There are exactly three solutions.

    (E) There are exactly four solutions.

    Solution: E. Draw two graphs of $y = |x^2 - 2x - 3|$ and $y = x + 2$, and count the number of intersection points of the graphs. Alternatively, square both sides of the equation to get $(x^2 - 2x - 3)^2 - (x + 2)^2 = 0$, and factor it to get $(x^2 - x - 1)(x^2 - 3x - 5) = 0$. Each quadratic equation has two real solutions, and therefore the equation has four real solutions. Checking to be sure that there are no extraneous roots, we find that there are indeed four solutions.

21. Let $a, b, c \geq 2$ be natural numbers and

    $$a^{(b^c)} = (a^b)^c?$$

    Which one(s) of a, b, c can have arbitrary values.

    (A) a **(B)** b **(C)** c **(D)** both a and b **(E)** all three

    Solution: A. The given equation is equivalent to $b^c = bc$ with a arbitrary. Thus

    $$b^{c-1} = c$$

    With natural $b, c \geq 2$, one solution is $b = c = 2$, but $a \geq 2$ is arbitrary.

22. Suppose $\log_8 a + \log_8 b = \log_8 a \cdot \log_8 b$ and $\log_a b = 3$. What is the value of a?

    (A) 9 **(B)** b^3 **(C)** 16

    (D) 32 **(E)** There is not enough information to answer

    Solution: C. If $\log_a b = 3$, then $(\log_8 b)/(\log_8 a) = 3$ by the base conversion property of logs. Simplifying gives $(\log_8 b) = 3(\log_8 a)$. By substitution into the original equation, $\log_8 a + 3\log_8 a = 3(\log_8 a)^2$. So $4\log_8 a = 3(\log_8 a)^2$. Since $\log_8 a$ is non-zero, dividing both sides by $\log_8 a$ gives $4 = 3(\log_8 a)$ and $4/3 = \log_8 a$. Solving for a gives $a = 8^{(4/3)} = 2^4 = 16$.

23. A movie was so awful that one-half of the audience left after a few minutes. Five minutes later, one-third of the remaining audience left. Ten minutes later, one-fourth of those remaining left, leaving only nine people in the audience. Let N denote the number of people in the audience at the beginning of the movie? Then

    (A) $N \in [1, 20]$ **(B)** $N \in [21, 30]$ **(C)** $N \in [31, 40]$

    (D) $N \in [41, 50]$ **(E)** $N > 50$

    Solution: C. Nine people were three-fourths of the audience before the last people left, so before the last exit, twelve people were in the audience. Those twelve people were two-thirds of the people after the first exit, so eighteen people were present after the first exit. Those eighteen were one-half of the total audience, so the movie began with an attendance of 36. Alternatively, the number left at the end is $(N \cdot \frac{1}{2})(\frac{2}{3})(\frac{3}{4}) = 9$ which reduces to $\frac{1}{4}N = 9$, so $N = 36$.

24. The four angles of a quadrilateral are in arithmetic progression and the largest is twice the smallest. What is the largest angle?

    (A) $60°$ **(B)** $90°$ **(C)** $100°$ **(D)** $120°$ **(E)** $140°$

    Solution: D. If the angles are denoted a, b, c, and d, respectively, then $a + a + e + a + 2e + a + 3e = 360$, where e is the common difference. It is given that $a + 3e = 2a$, which reduces to $a = 3e$. Using this with the equation above yields $e = 20$ from which it follows that $a = 60, b = 80, c = 100,$ and $d = 120$.

25. The sum of three numbers is 155. The same number k is obtained using any of the three operations below:

    a. Seven is added to the smallest number.

    b. Seven is subtracted from the middle number.

    c. The largest number is divided by 3.

    How much greater is the largest than the smallest?

    (A) 57 **(B)** 63 **(C)** 65 **(D)** 67 **(E)** 69

    Solution: E. If a, b, and c are the numbers in increasing order, then $a + 7 = b - 7 = c \div 3$, $b = a + 14$, $c = 3a + 21$, and it follows that $a + a + 14 + 3a + 21 = 155$. This yields $5a + 35 = 155$ or $a = 24$. Then $b = 38$ and $c = 93$, so the difference is $93 - 24 = 69$.

    Alternatively, using $a < b < c$, $a + 7 = b - 7 = c/3$. Thus $a + b = (a + 7) + (b - 7) = 2c/3$. Therefore, $155 = a + b + c = 5c/3$ and we obtain $c = 93$ and from this $c - a = c - (c/3 - 7) = 2c/3 + 7 = 69$.

26. A standard deck of 52 cards has four *suits*, clubs, diamonds, hearts, and spades. Each suit has 13 *values* including ace, two, three, ..., ten, and three face cards, Jack, Queen, and King. What is the fewest number of cards that must be selected from a deck to guarantee that the set contains three-of-a-kind; that is, three cards of the same value?

    (A) 15 **(B)** 18 **(C)** 20 **(D)** 25 **(E)** 27

    Solution: E. You must draw 27 cards to be sure to get a three-of-a-kind because the first 26 cards you select might consist of 13 pairs.

27. Let $N = \underline{abcde}$ denote the five digit number with digits a, b, c, d, e and $a \neq 0$. Let $N' = \underline{edcba}$ denote the reverse of N. Suppose that $N > N'$ and that $N - N' = 5x014$ where x is a digit. What is x?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    Solution: E. The sum of the digits of N is the same as the sum of the digits of N', so $N - N'$ is a multiple of 9. Thus, the remainder when $5 + x + 1 + 4 = 10 + x$ is divided by 9 is zero, ie, $x = 8$.

    Alternatively, since $N - N' > 0$ and the units digit of $N - N'$ is not 0 (or because the difference is another 5 digit number), $e < a$. Combining this with the fact that the middle digit of $N - N'$ is 0, implies $d - 1 \geq b$ and so $d - 1 - b = 1$. Thus $d - b = 2$ and therefore $10 + b - d = 8 = x$. The original number could have been 74261, for example, since $74261 - 16247 = 58014$.

28. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

    (A) 1 **(B)** 4 **(C)** 36 **(D)** 49 **(E)** 81

    Solution: A. The terms of the arithmetic progression are 9, $9+d$, and $9+2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

    $$(11+d)^2 = 9(29+2d) \quad \text{so} \quad d^2 + 4d - 140 = 0.$$

    Thus $d = 10$ or $d = -14$. The corresponding geometric progressions are 9, 21, 49 and 9, -3, 1, so the smallest possible value for the third term of the geometric progression is 1.

29. Betty has 6 daughters and no sons. Some of her daughters have 6 daughters, and the rest have none. Betty has a total of 30 daughters and granddaughters, and no great-granddaughters. How many of Betty's daughters and granddaughters have no daughters?

    (A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

    Solution: C. Betty has $30 - 6 = 24$ granddaughters, none of whom have any daughters. The granddaughters are the children of $24/6 = 4$ of Betty's daughters, so the number of women having no daughters is $30 - 4 = 26$.

30. How many digits are required to represent the number 2007^{2007} in decimal form?

    (A) 6628 **(B)** 6629 **(C)** 6630 **(D)** 6631 **(E)** 7321

    Solution: B. The one digit numbers, 1 through 9, all have common logarithm value in the interval $[0,1)$. Two digit numbers have common logarithm value between 1 and 2. In general, for a positive integer N, N has $k+1$ digits if the common logarithm of N satisfies $k \le \log N < k+1$). Since $\log(2007^{2007}) = 2007 \cdot \log(2007) = 6628.2125\ldots$, it follows that 2007^{2007} has 6629 digits.