UNC Charlotte 2002 Algebra with Solutions

March 4, 2002

1. Suppose A and B are sets with 5 and 7 elements respectively and $A \cap B$ has 2 elements. How many elements does $A \cup B$ have?

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

   (C) The inclusion-exclusion principle states that for any pair of sets, A and B, $|A \cup B| = |A| + |B| - |A \cap B|$, which in this case yields $|A \cup B| = 5 + 7 - 2 = 10$.

2. Find x such that
$$\frac{8+x}{8-x} = \frac{x}{x+x}.$$

   (A) $-8/3$ **(B)** $-4/3$ **(C)** $-2/3$ **(D)** 0 **(E)** 4

   (A) The equation is equivalent to $2(8+x) = 8 - x$, from which it follows that $x = -8/3$.

3. What is the length of the interval of solutions to the inequality $1 \leq 3 - 4x \leq 9$?

   (A) 1.75 **(B)** 2.00 **(C)** 2.25 **(D)** 2.50 **(E)** 3.25

   (B) Subtract 3 from all parts to get $-2 \leq -4x \leq 6$, then divide all by -4 to get $1/2 \geq x \geq -3/2$, so the length of the interval is $1/2 - (-3/2) = 2$.

4. Find the values of x for which $x^2 + 3x - 4 > 0$.

   (A) $x < 1$ and $x > -4$ **(B)** $x > 1$ or $x < -4$ **(C)** $x < -1$ or $x > 4$
   (D) $x > -1$ and $x < 4$ **(E)** none of **A**, **B**, **C**, or **D**

   (B) Factor to find the zeros of $x^2 + 3x - 4 = (x+4)(x-1)$. Inspection of the intervals determined by $x = -4$ and $x = 1$ yields the two intervals $x > 1$ or $x < -4$.

5. The equation $ax^2 - 2x\sqrt{2} + 1 = 0$ has a zero discriminant, where a is a real number. Find the root(s) of the equation.

   (A) $\pm\sqrt{2}$ **(B)** 1 **(C)** 2 **(D)** $\pm\frac{\sqrt{2}}{2}$ **(E)** $\frac{\sqrt{2}}{2}$

   (E) Note that $b^2 - 4ac = 8 - 4a = 0$, so $a = 2$. Thus $x = \frac{2\sqrt{2}\pm 0}{2(2)} = \sqrt{2}/2$.

6. It is known that the equation $ax^2 + 5x = 3$ has a solution $x = 1$. Find the other solution.

   (A) 0.5 **(B)** 1.5 **(C)** 2 **(D)** 2.5 **(E)** 3

   (B) Replace x by 1 we have $a + 5 = 3$, so $a = -2$. Solving the new equation $-2x^2 + 5x = 3$, we obtain the other solution $x = 1.5$

7. Let x_1, x_2 be the two solutions to the equation $2x^2 - x - 2 = 0$. Find the value of $\frac{1}{x_1} + \frac{1}{x_2}$.

   (A) -3 **(B)** -2 **(C)** -1 **(D)** $-1/2$ **(E)** $1/2$

   (D) Note that $\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1+x_2}{x_1x_2} = \frac{\frac{1}{2}}{-1} = -\frac{1}{2}$.

8. It is known that $x = 1$ is a solution to the equation $2x^3 - 3x^2 - 4x + 5 = 0$. What are the two other solutions?

   (A) $\dfrac{-1 \pm \sqrt{41}}{4}$ **(B)** $\dfrac{1 \pm \sqrt{41}}{4}$ **(C)** $\dfrac{1 \pm \sqrt{39}}{4}$ **(D)** $\dfrac{1 \pm \sqrt{41}}{2}$ **(E)** ± 1

   (B) Note that $(x - 1)$ is a factor of the polynomial $2x^3 - 3x^2 - 4x + 5$. After long division we have $(x - 1)(2x^2 - x - 5) = 0$. Now the other two solutions are obtained from the quadratic formula.

9. It takes 852 digits to number the pages of a book consecutively. How many pages are there in the book?

   (A) 184 **(B)** 235 **(C)** 320 **(D)** 368 **(E)** 425

   (C) Pages 1 through 9 use 9 digits and 10 through 99 use $90 \times 2 = 180$ digits, for a total of 189 digits for pages 1 through 99. That leaves 663 digits remaining to make the required total of 852 digits. These are obtained by going 221 pages beyond page 99, through page 320.

10. Solve the equation $8^{\frac{1}{6}} + x^{\frac{1}{3}} = \frac{7}{3-\sqrt{2}}$.

    (A) 24 **(B)** 27 **(C)** 32 **(D)** 64 **(E)** none of **A**, **B**, **C** or **D**

    (B) Note that $8^{\frac{1}{6}} = (2^3)^{\frac{1}{6}} = 2^{\frac{1}{2}}$. Rationalizing the denominator of the right side gives $\frac{7}{3-\sqrt{2}} = \frac{7(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{21+7\sqrt{2}}{7} = 3 + \sqrt{2}$. Thus the equation reduces to $x^{\frac{1}{3}} = 3$ or $x = 27$. Alternatively, you can use a calculator to solve this problem.

11. The fraction $\frac{5x-11}{2x^2+x-6}$ was obtained by adding the two fractions $\frac{A}{x+2}$ and $\frac{B}{2x-3}$. Find the value of $A+B$.

    (A) -4 **(B)** -2 **(C)** 1 **(D)** 2 **(E)** 4

    (D) Note that $\frac{5x-11}{2x^2+x-6} = \frac{A}{x+2} + \frac{B}{2x-3} = \frac{A(2x-3)}{(x+2)(2x-3)} + \frac{B(x+2)}{(x+2)(2x-3)}$. Also, note that $5x - 11 = (2A + B)x - (3A - 2B)$, so $2A + B = 5$ and $3A - 2B = 11$. Solving this system of equations we obtain $A = 3$ and $B = -1$, so $A + B = 2$. Alternatively, since the degree of the numerator is less than the degree of the denominator, the value for A can be obtained by evaluating the expression $\frac{5x-11}{2x-3}$ at $x = -2$ (the zero of $x + 2$) and the value for B can be obtained by evaluating the expression $\frac{5x-11}{x+2}$ at $x = \frac{3}{2}$ (the zero of $2x-3$). So $A = \frac{-21}{-7} = 3$, $B = \frac{-(7/2)}{(7/2)} = -1$, and $A + B = 2$.

12. The slope of the line through the points that satisfy $y = 8 - x^2$ and $y = x^2$ is

    (A) 2 **(B)** 4 **(C)** 0 **(D)** -2 **(E)** -4

    (C) The slope is 0 because the two parabolas are intersecting even functions. In fact they are reflections (through the line $y = 4$) of each other. Alternatively, $8 - x^2 = x^2 \Rightarrow x = \pm 2, y = 4$, and the slope of the line $y = 4$ is 0.

13. The product of the zeros of $f(x) = (2x - 24)(6x - 18) - (x - 12)$ is

    (A) -72 **(B)** 5 **(C)** 6 **(D)** 37 **(E)** 432

    (D) The roots are 12 and 37/12 because $f(x) = 12 \cdot (x-12) \cdot (x-3) - (x-12) = (x - 12)(12x - 37)$ so the product is $12 \cdot (37/12) = 37$.

14. Factor $x^4 + 4y^4$ over the real numbers. Hint: Add and subtract $4x^2y^2$.

    (A) $(x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$

    (B) $(x^2 + 2xy + 2y^2)^2$

    (C) $(x^2 + 2xy - 2y^2)(x^2 + 2xy + 2y^2)$

    (D) $(x^2 - 2xy - 2y^2)(x^2 + 2xy + 2y^2)$

    (E) none of **A**, **B**, **C**, or **D**

(A) Add and subtract $4x^2y^2$ to get $x^4 + 4y^4 = x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 = (x^2 + 2y^2)^2 - (2xy)^2 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$.

15. What is the remainder when $x^2 + 3x - 5$ is divided by $x - 1$?

    (A) -5 **(B)** -2 **(C)** -1 **(D)** 0 **(E)** 1

(C) By long division or the Remainder Theorem, the remainder is -1.

16. Jeremy starts jogging at a constant rate of five miles per hour. Half an hour later, David starts running along the same route at seven miles per hour. For how many minutes must David run to catch Jeremy?

    (A) 75 minutes **(B)** 80 minutes **(C)** 90 minutes

    (D) 95 minutes **(E)** 105 minutes

(A) David runs at 7 mph for t hours while Jeremy runs for $t + 1/2$ hours at 5 mph. They run the same distance, so $5 \cdot (t + 1/2) = 7t$, which yields $t = 5/4$ hours, or 75 minutes.

17. For the final exam in Professor Ahlin's class, the average(= arithmetic mean) score of the group of failing students was 62 and the average score among the passing students was 92. The overall average for the 20 students in the class was 80. How many students passed the final?

    (A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

    (D) Translating the information into equations, where x represents the number of students who passed the final,

    $$62(20 - x) + 92(x) = 20 \cdot 80 = 1600.$$

    Solve this for x to get $x = 12$. Alternatively, since the difference between 92 and 62 is 30 and the difference between 92 and 80 (the average) is 12 while that between 80 and 62 is 18, $80 = \frac{18}{30} \cdot 92 + \frac{12}{30} \cdot 62$. Moreover, the same combination must give the total number of students in the class. So $\frac{18}{30} \cdot 20 = 12$ is the number of students who passed and $\frac{12}{30} \cdot 20 = 8$ is the number who failed.

18. Fifteen numbers are picked from the set $\{1, 2, 3, \ldots 20, 21\}$. Find the probability that at least three of those numbers are consecutive.

    (A) 0.1 **(B)** 0.2 **(C)** 0.4 **(D)** 0.5 **(E)** 1.0

    (E) Imagine putting the 15 numbers into seven boxes labeled 123, 456, 789, etc. Each number is put into the box that it helps to label. After all 15 numbers have been distributed among the boxes, some box must have three balls, by the Pigeon-Hole Principle. Thus the probability that some box has three consecutive numbers is 1.

19. Cara has 162 coins in her collection of nickels, dimes, and quarters, which has a total value of \$22.00. If Cara has twelve fewer nickels than quarters, how many dimes does she have?

    (A) 50 **(B)** 60 **(C)** 70 **(D)** 74 **(E)** 78

    (C) Solve simultaneously the three equations $q - 12 = n$, $n + d + q = 162$ and $5n + 10d + 25q = 2200$ to get $d = 70$.

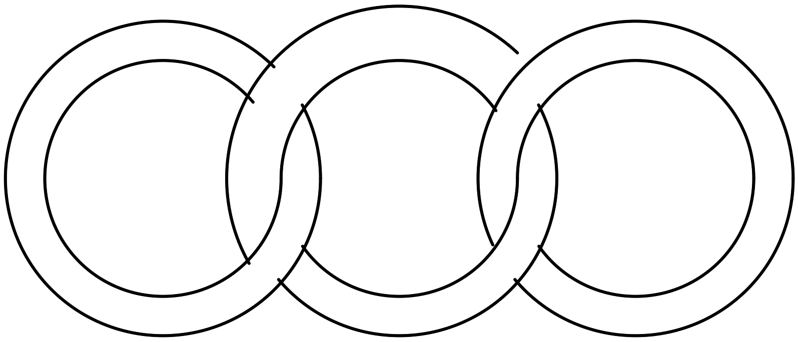
20. Let N denote the smallest four-digit number with all different digits that is divisible by each of its digits. What is the sum of the digits of N?

    (A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

    (D) Trying the smallest digits first, we are led to numbers of the form $12xy$, which in turn leads to 1236.

21. A chain with two links is 13 cm long. A chain made from three links, as shown, of the same type is 18 cm long. How long is a chain made from 25 such links?

    (A) 120 **(B)** 128 **(C)** 136 **(D)** 144 **(E)** 150



(B) Look at the diagram, and note the length of a chain with n links is $2R + 2(n-1)r$ where R is the outer radius and r is the inner radius. Thus, $2R+2r = 13$ and $2R+4r = 18$, which yields $2r = 5/2$ and $2R = 8$. Thus the length of a 25 link chain is $2R+48r = 8+120 = 128$. Alternatively, note that each link adds 5 cm. to the length of the chain, for a total of $8+5\cdot 24 = 128$.

22. What is the sum of the three positive integers a, b, and c that satisfy

$$\frac{1}{a+\frac{1}{b+\frac{1}{c}}} = 3/16?$$

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 11

(C) First note that $a = 5$ because $16/3 = 5\frac{1}{3} = a + \frac{1}{b+\frac{1}{c}}$ and that $\frac{1}{b+\frac{1}{c}} = 1/3$. So $b + \frac{1}{c} = 3$ and it follows that $b = 2$ and $c = 1$. Thus $a + b + c = 8$.

23. A circle C contains the points $(0, 6), (0, 10)$, and $(8, 0)$. What is the x-coordinate of the center?

(A) 6.75 **(B)** 7.25 **(C)** 7.50 **(D)** 7.75 **(E)** 8.25

(D) The center (h, k) of C must lie on the line $y = 8$, because the center is the same distance from $(0, 6)$ as it is from $(0, 10)$. Thus the circle satisfies $(x-h)^2 + (y-8)^2 = r^2$, for some number h. It must also lie on the line that perpendicularly bisects the segment from $(0, 6)$ to $(8, 0)$, an equation for which is $y - 3 = \frac{4}{3}(x-4)$. Since $y = 8$, it follows that $h = 7.75$. Alternatively, the distance from $(h, 8)$ to $(0, 6)$ must be the same as the distance from $(h, 8)$ to $(8, 0)$, which we can solve for h.

24. The number 839 can be written as $19q + r$ where q and r are positive integers. What is the largest possible value of $q - r$?

    (A) 37 **(B)** 39 **(C)** 41 **(D)** 45 **(E)** 47

    (C) Divide 839 by 19 to get $839 = 44 \cdot 19 + 3$. One pair of positive value is $q = 44$ and $r = 3$. If we try to make q any bigger, we are forced to make r negative.

25. Vic can beat Harold by one tenth of a mile in a two mile race. Harold can beat Charlie by one fifth of a mile in a two mile race. If Vic races Charlie, how far ahead will Vic finish?

    (A) 0.15 miles **(B)** 0.22 miles **(C)** 0.25 miles

    (D) 0.29 miles **(E)** 0.33 miles

    (D) Vic is 20/19 times as fast as Harold. Harold is 20/18 times as fast as Charlie, so Vic is $(20/19)(20/18) \approx 1.1696$ times as fast as Charlie. When Vic runs 2 miles, Charlie will have run $2/1.1696 \approx 1.71$ miles. Vic will finish 0.29 miles ahead of Charlie.

26. The four angles of a quadrilateral form an arithmetic sequence. The largest is 15 degrees less than twice the smallest. What is the degree measure of the largest angle?

    (A) 95° **(B)** 100° **(C)** 105° **(D)** 115° **(E)** 125°

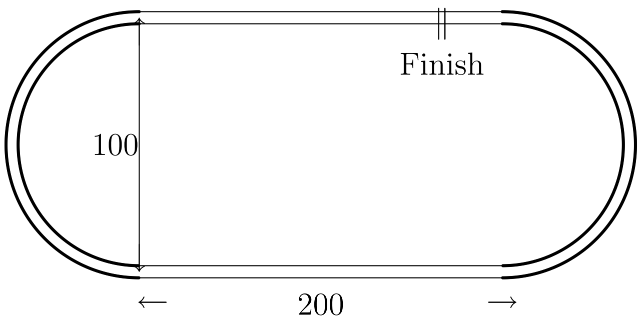
    (D) Let the angles be $a, a + d, a + 2d$, and $a + 3d$. Then $4a + 6d = 360$ and $a + 3d + 15 = 2a$. Thus $a = 3d + 15$, so we can replace a with $3d + 15$ in the first equation. Solve this to get $3d = 50$ and $a = 50 + 15 = 65$, so the largest angle is $65 + 50 = 115°$.

27. What is the probability of obtaining an ace on both the first and second draws from an ordinary deck of 52 playing cards when the first card is not replaced before the second is drawn? There are four aces in such a deck.

    (A) 1/221 **(B)** 4/221 **(C)** 1/13 **(D)** 1/17 **(E)** 30/221

    (A) The probability of obtaining an ace on the first draw is $4/52 = 1/13$. If the first card drawn is an ace there are 3 aces remaining in the deck, which now consists of 51 cards. Thus, the probability of getting an ace on the second draw is $3/51 = 1/17$. The required probability is the product of the two, which is $1/221$.

28. A running track has the shape shown below. The ends are semicircular with diameter 100 yards. Suppose that the lanes are each 1 yard wide and numbered from the inside to the outside. The competitor in the inside lane runs 700 yards counter clockwise. The other runners start ahead of the inside lane runner, and also run 700 yards, with all runners finishing at the same place. Approximately how much of a head start should a runner in the fifth lane receive over a runner in the first lane?



(A) 15 yards **(B)** 20 yards **(C)** 25 yards **(D)** 30 yards **(E)** 35 yards

(C) In order to run 700 yards, the runners must traverse both semicircular ends. The runner in the first lane has an inner radius of 50 yards, while the runner in the fifth lane has an inner radius of 54 yards. The difference in distance is $2\pi(54-50) = 8\pi \approx 25$ yards.

29. Dick and Nick share their food with Albert. Dick has 5 loaves of bread, and Nick has 3 loaves. They share the bread equally. Albert gives Dick and Nick 8 dollars which they agree to share fairly. How should they divide the \$8 between them?

(A) Dick should get \$3 of Albert's money. **(B)** Dick should get \$4 of Albert's money.

(C) Dick should get \$5 of Albert's money. **(D)** Dick should get \$6 of Albert's money.

(E) Dick should get \$7 of Albert's money.

(E) Albert pays \$8 for his 8/3 loaves, so loaves must be worth \$3 each. Nick eats all but 1/3 of a loaf of his bread while Dick gives up 7/3 loaves. Thus Dick should get \$7 of Albert's money.

30. For what positive value of x is there a right triangle with sides $x+1, 4x$, and $4x+1$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

(B) The number x must satisfy $(x+1)^2 + (4x)^2 = (4x+1)^2$ since $4x+1$ is certainly the largest of the three numbers. Thus $x^2 + 2x + 1 + 16x^2 - 16x^2 - 8x - 1 = x^2 - 6x = 0$ from which it follows that $x = 6$.