**Final Exam** **Math 6105**

**July 1, 2005** Your name ______________________________

Throughout this test you must **show your work**.

1. (8 points) Use the repeated subtraction method to find the base 5 representation of each of the following numbers

   (a) 93

   (b) 17.24

2. (8 points) Use the method of repeated multiplication to find a base 5 representation of each of the following numbers

   (a) 0.275

   (b) 29/125

3. (8 points) Find the base $-5$ representation of each of the following numbers

   (a) 346

   (b) 17.2

4. (8 points) Find the Fibonacci representation of each of the following numbers

   (a) 127

   (b) 188

5. (8 points) You're playing the dynamic doubling nim $N_d(k)$ and your opponent has just left you the position $(127, 40)$. How many moves to safe positions do you have? What are they? What are the safe positions you can leave for your opponent?

6. (18 points) Consider the game of Bouton's nim with pile sizes $19, 24, 25, 27, 37$.

   (a) Find the binary representation of each pile size.

   (b) Find the binary configuration of the game. That is, write these binary numbers in a column and compute their nim sum.

   (c) Notice that the binary configuration is not balanced since the nim sum of the pile sizes is not zero. Find a move which results in a balanced binary configuration. Is there just one such move or are there several?

   (d) Suppose you made a move which balances the configuration. Assume your opponent takes five counters from the same pile as the one from which you removed counters. What move do you make now?

7. (15 points) Solve the decanting problem for containers of sizes 199 and 179; that is find integers $x$ and $y$ satisfying

$$199x + 179y = d$$

   where $d$ is the GCD of 199 and 179.

8. (15 points) Find digits $a$, $b$, and $c$ (between 0 and 4) such that $abc_5 = cba_8$, or prove that there are none.

9. (10 points) Use the Principle of Mathematical Induction to prove that for every positive integer $n$, $9^n - 4^n$ is divisible by 5.

10. (15 points) The sequence of Fibonacci numbers $f_0, f_1, f_2, \ldots$ is defined by the rule $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$ for all $n \geq 0$.

11. (20 points) Suppose $S = \{1, 2, \ldots, 20\}$. A subset $T$ of size 3 is randomly selected.

   (a) Find the probability that $T$ consists of two odd numbers and one even number.

   (b) Find the probability that $T$ consists of three prime numbers.

   (c) Find the probability that the three numbers in $T$ have a sum that is less than 9.

   (d) Find the probability that $T$ has at least one even number in it.

12. (35 points) A discrete math class has 10 women and 7 men.

   (a) How many 5 element subsets does the class have?

   (b) How many ways are there to choose a committee of size 5 consisting entirely of women?

   (c) How many ways are there to choose a committee of size 5 consisting of 4 women and 1 man?

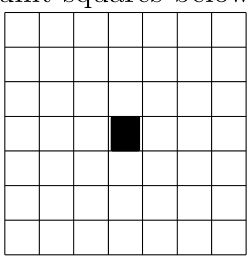
   (d) How many ways are there to choose a committee of size 5 consisting of 3 women and 2 men?

   (e) How many ways are there to choose a committee of size 5 consisting of 2 women and 3 men?

   (f) How many ways are there to choose a committee of size 5 consisting of 1 woman and 4 men?

   (g) How many ways are there to choose a committee of size 5 consisting of 5 men?

13. (20 points) Consider the grid of unit squares below.



(a) How many square regions are bounded the grid lines?

(b) How many rectangular regions are bounded by grid lines?

(c) How many square regions containing the shaded square are bounded the grid lines?

(d) How many rectangular regions containing the shaded square are bounded by grid lines?

14. (35 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Its important to recall that subsets have no order; that is, for example $\{1, 2, 3\} = \{2, 3, 1\}$, but numbers change when the order of their digits changes; ie, $123 \neq 231$.

    (a) How many non-empty subsets does $S$ have?

    (b) How many subsets of $S$ have no even numbers as members?

    (c) How many subsets of $S$ have exactly 4 elements?

    (d) How many four-element subsets of $S$ contain exactly two odd numbers?

    (e) How many four-digit numbers can be made using the digits of $S$ if a digit may be used repeatedly?

    (f) How many four-digit numbers can be made using the digits of $S$ if a digit may be used only once?

    (g) How many even four-digit numbers bigger than 3000 can be made using the digits of $S$ if a digit may be used only once?

15. (20 points) Recall that if $R$ is an equivalence relation on a set $A$, the set $\{[x] | x \in A\}$ is a partition of $A$.

    (a) Let $A = \{1, 2, 3, 4, 5\}$ and define $xRy$ to mean that $x + y$ is even. Show that $R$ is an equivalence relation, and find the partition induced on $A$ by $R$.

    (b) How many equivalence relations are there on the set $A = \{1, 2, 3, 4, 5\}$?

16. (20 points) Let $R$ be the following relation defined on the set $\{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)\}.$$

Determine whether $R$ is:

(a) reflexive

(b) symmetric

(c) antisymmetric

(d) transitive

17. (15 points) Suppose $A, B,$ and $C$ are sets of integers such that $B \cap C = \phi, |B| = 15, |C| = 12, |A \cap B| = |A \cap C| = 2,$ and $|A \cup B| = 28$. Find each of the following:

(a) $|A \cap \overline{C}|$

(b) $|(A \cup B) \cap \overline{C}|$

(c) $|A \cup B \cup C|$

18. (25 points) Let $A = \{1, 2, 3\}$.

(a) Give an example of a $3 \times 3$ boolean matrix which represents a relation on $A$ that is both symmetric and antisymmetric. Which entries of the matrix can be either 0 or 1?

(b) Use the information in (a) to count the number of relations on $A$ which are both symmetric and antisymmetric. (Remember that there are $2^9$ relations on $A$).

(c) How many relations on $A$ are antisymmetric?

(d) How many relations on $A$ are symmetric?

19. (20 points) Let $A = \{1, 2, 3, 4\}$. Find examples of relations on $A$ which satisfy each of the following collections of conditions:

    (a) Reflexive and symmetric and not transitive.

    (b) Reflexive and antisymmetric and not transitive.

    (c) Symmetric and transitive and not antisymmetric.

    (d) Symmetric and not transitive and not reflexive.