Math 1242 **Calculus** **Test 3**

November 27, 2005 **Name** ______________________________

On all the following questions, **show your work.** There are 139 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (10 points) Suppose the series $\sum a_n$ has partial sums S_n given by $S_n = \frac{(2n-1)^2}{(3n+1)^2}$. Does the series converge? If so, to what?

   Solution: By the horizontal asymptote theorem, the limit as $n \to \infty$ is $4/9$.

2. (20 points) Test for convergence and find the sum if possible. If you cannot find the sum, state the test you used to determine convergence (or divergence).

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

   Solution: Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the series truncates, yielding $S_n \to 1$.

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

   Solution: This is just the Maclaurin series for e^x at $x = 2$, so $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + (\pi/e)^n}$.

   Solution: This converges by the comparison test, comparing with $\displaystyle\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$, which is geometric.

   (d) $\displaystyle\sum_{n=1}^{\infty} \sin(n+1) - \sin(n)$.

   Solution: This truncates, but $\lim \sin(n+1)$ does not exist.

   (e) $\displaystyle\sum_{n=1}^{\infty} \arctan(n+1) - \arctan n$.

   Solution: This truncates, and $\displaystyle\lim_{n\to\infty} \arctan(n+1) - \arctan 1 = \pi/2 - \pi/4 = \pi/4$.

3. (25 points) Match each of the following with the correct statement.

   A. The series is absolutely convergent.

   C. The series converges, but is not absolutely convergent.

   D. The series diverges.

   ___1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+4}$

   ___2. $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2}$

   ___3. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+7}$

   ___4. $\sum_{n=1}^{\infty} \frac{(n+1)(6^2-1)^n}{6^{2n}}$

   ___5. $\sum_{n=1}^{\infty} \frac{(-5)^n}{n^5}$

   Solution:

   1. C, compare with harmonic. $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+4}$
   2. A, compare with $\sum 1/n^2$. $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2}$
   3. C, compare with $sum 1/\sqrt{n}$. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+7}$
   4. A, almost geometric, use ratio test. $\sum_{n=1}^{\infty} \frac{(n+1)(6^2-1)^n}{6^{2n}}$
   5. D, by ratio test. $\sum_{n=1}^{\infty} \frac{(-5)^n}{n^5}$

4. (24 points) The interval of convergence of a power series can be of four forms, $[a, b], (a, b], [a, b)$ and (a, b). For each part below gave an example of a power series with the given interval of convergence.

(a) $(0, 2)$

Solution: $\sum_{n=0}^{\infty}(x-1)^n$

(b) $[-1, 5]$

Solution: $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2 \cdot 3^n}$.

(c) $[1, 7)$

Solution: $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n \cdot 3^n}$

5. (20 points) Consider the function $f(x) = e^{2x-1}$.

(a) Find the Taylor polynomial $T_5(x)$ at $a = 1/2$.

Solution: The n^{th} derivative $f^{(n)}$ of f is given by $f^(n)(x) = 2^n e^{2x-1}$, so $f^{(n)}(1/2) = 2^n e^{2(1/2)-1} = 2^n$. Therefore $c_n = 2^n/n!, n = 0, 1, \ldots$. Thus, $T_5(x) = 1+2(x-1/2)+4(x-1/2)^2/2+8(x-1/2)^3/6+16(x-1/2)^4/24+32(x-1/2)^5/120$. Alternatively, recall that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. Replace the x with $2x - 1$ to get the Taylor series for $f(x)$.

(b) Find an upper bound for $|R_5(x)|$ on the interval $[0, 1]$.

Solution: Use Taylor's Inequality Theorem with $n = 5$ and $a = d = 1/2$ to get $M = 2^6 e$ and finally $|R_5(x)| \le 2^6 e/6! \times (1/2)^6 = 1/720$.

(c) Find the radius of convergence of the Taylor series.

Solution: The ratio test yields $2|x|/(n+1) \to 0$ as $n \to \infty$, so the series converges for all x.

6. (20 points) Consider the function $f(x) = \frac{1}{1-x^2}$.

(a) Find a power series representation of $f(x) = \frac{1}{1-x^2}$.

Solution: Recall that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$. Replace x by x^2 to get $f(x) = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdot = \sum_{n=0}^{\infty} x^{2n}$

(b) Differentiate both sides of the equation in (a) to find a power series representation of $f'(x)$ and find the interval of convergence for this series.

Solution: First differentiate f to get $f'(x) = -(1-x^2)^{-2} \cdot -2x = \frac{2x}{(1-x^2)^2}$. Then note that we can differentiate the series term by term to get $\frac{2x}{(1-x^2)^2} = \sum_{n=0}^{\infty} 2nx^{2n-1}$.

7. (20 points) Consider the function $f(x) = 3x^2\sin(x^2)$. Recall that the Maclaurin series for $\sin x$ is given by $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

   (a) Find the Maclaurin series representation of $f(x)$.

   Solution: The Maclaurin series representation of $\sin x$ is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$. Replace x with x^2 to get the series for $\sin(x^2)$ is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!}$ and multiply this by $3x^2$ to get $3x^2\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}3(-1)^n\frac{(x^2)^{2n+2}}{(2n+1)!}$.

   (b) Use part (a) of the problem to find each of the following derivatives of f.

   i. $f^{(3)}(0)$
   Solution: We have two ways to look at the series for $f(x) = 3x^2\sin(x^2)$. On one hand, the formula gives $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$. On the other hand $f(x) = \displaystyle\sum_{n=1}^{\infty}3(-1)^n\frac{(x^2)^{2n+2}}{(2n+1)!}$. The term involving $f'''(0)$ in the formula is $f^{(3)}(0)/3!\times x^3$, so we need to find the term of $\displaystyle\sum_{n=1}^{\infty}3(-1)^n\frac{(x^2)^{2n+2}}{(2n+1)!}$ that gives us an x^3. Since $(x^2)^{2n+2} = x^{4n+4}$, we can see that only for multiples of 4 do we get nonzero derivatives. Thus $f'''(0) = 0$.

   ii. $f^{(4)}(0)$
   Solution: As above we have $f^{(4)}(0)/4! \times x^4 = 3\frac{(x)^{4\cdot 0+4}}{(2\cdot 0+1)!} = x^4$, so $f^(4)(0) = 3\cdot 4! = 72$.

   iii. $f^{(8)}(0)$
   Solution: Here we get $\frac{f^{(8)}(0)}{8!} = -3/3!$ and this leads to $f^{(8)}(0) = -20160$.

   iv. $f^{(12)}(0)$
   Solution: Here we get $\frac{f^{(12)}(0)}{12!} = 3/5!$ and this leads to $f^{(12)}(0) = \frac{3\cdot 12!}{5!} = 11,975,040$.