Tuesday, April 26, 2005 **Name** ____________________________

On all the following questions, **show your work.** There are 155 points available on this test. Do not try to do all the problems. Try to find four or five that you can do well.

1. (5 points) What is the limit of the *sequence* a_n defined by $a_n = \frac{n^2-6n+5}{3n^2+2n-12}$, as $n \to \infty$?

   Solution: Divide both numerator and denominator by n^2 to get

   $$\lim_{n\to\infty} \frac{n^2-6n+5}{3n^2+2n-12} = \lim_{n\to\infty} \frac{n^2/n^2 - 6n/n^2 + 5/n^2}{3n^2/n^2 + 2n/n^2 - 12/n^2}$$
   $$= \lim_{n\to\infty} \frac{1 - 6n/n^2 + 5/n^2}{3 + 2n/n^2 - 12/n^2} = 1/3.$$

2. (15 points) Consider the series $\sum\limits_{n=1}^{\infty} a_n$ where

   $$a_n = \frac{(6n^2+4)3^{n+1}}{5^n}$$

   Does the series converge absolutely, converge conditionally or diverge. What test did you use? Why is it conclusive?

   Solution: Using the ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3/5$, so the series converges absolutely.

3. (15 points) Consider the series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \frac{e^{n-3}}{\sqrt{n+5}(n+2)!}$$

Does the series converge absolutely, converge conditionally or diverge. What test did you use? Why is it conclusive?

Solution: Using the ratio test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, so the series converges absolutely.

4. (20 points) Find the interval of convergence for the given power series.

$$\sum_{n=1}^{\infty} \frac{(x-8)^n}{n(-6)^n}$$

(a) What is the radius of convergence?

Solution: Using the ratio test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-8)}{-6}\right| = \left|\frac{(x-8)}{6}\right|$, so the series converges when $|x-8| < 6$. The radius of convergence is $R = 6$.

(b) Discuss convergence at the left endpoint.

Solution: The left endpoint is 2. Thus $\Sigma_{n=1}^{\infty} \frac{(2-8)^n}{n(-6)^n} = \Sigma_{n=1}^{\infty} \frac{1}{n}$ which diverges.

(c) Discuss convergence at the right endpoint.

Solution: The right endpoint is 14, and $\Sigma_{n=1}^{\infty} \frac{(14-8)^n}{n(-6)^n} = \Sigma_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test.

(d) What is the interval of convergence?

Solution: Putting parts b. and c. together, the interval of convergence is $(2, 14]$.

5. (20 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(7x)^n}{n^{11}}$$

(a) What is the radius of convergence?

Solution: Using the ratio test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |7x|$, so the series converges when $|7x| < 1$. The radius of convergence is $R = 1/7$.

(b) Discuss convergence at the left endpoint.

Solution: At $x = -1/7$, the series is $\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n^{11}}$, which converges by the alternating series test.

(c) Discuss convergence at the right endpoint.

Solution: At $x = 1/7$, the series is $\Sigma_{n=1}^{\infty} \frac{1}{n^{11}}$, which is a p series with $p > 1$, so it converges.

(d) What is the interval of convergence?

Solution: $[-1/7, 1/7]$.

6. (20 points) Find all the values of x such that the given series would converge.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{4^n}$$

(a) What is the radius of convergence?

Solution: Again, using the ratio test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-4}{4}\right|$, so the series converges when $|x-4| < 4$. The radius of convergence is $R = 4$.

(b) Discuss convergence at the left endpoint.

Solution: At $x = 0$, the series is $\Sigma_{n=1}^{\infty} \frac{(0-4)^n}{4^n} = \Sigma_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+\ldots$. It diverges.

(c) Discuss convergence at the right endpoint.

Solution: At $x = 8$, the series is $\Sigma_{n=1}^{\infty} \frac{(8-4)^n}{4^n} = \Sigma_{n=1}^{\infty} 1 = 1+1+1+1+1+\ldots$. It diverges.

(d) What is the interval of convergence?

Solution: $(0, 8)$.

7. (20 points) Suppose that the Maclaurin series for $f(x) = \dfrac{5x}{(7+x)}$ is $\Sigma_{n=0}^{\infty} c_n x^n$. Find the coefficients, c_0, c_1 , c_2, c_3, and c_4. Find the radius of convergence R of the power series.

   Solution: First note that $f(x) = \dfrac{5x}{(7+x)} = \dfrac{5x}{7} \cdot \dfrac{1}{(1-(-x/7))}$. Since $\frac{1}{1-x} = \Sigma_{n=0}^{\infty} x^n$, it follows that $f(x) = \frac{5x}{7} \Sigma_{n=0}^{\infty} (-x/7)^n$. The radius of convergence of this series is $R = 7$ and the first few values of c_n are $c_0 = 0, c_1 = 5/7, c_2 = -5/49, c_3 = 5/243$ and $c_4 = -5/2401$.

8. (10 points) Find the Maclaurin series of the function $f(x) = (9x^2)e^{-x}$. That is, find c_0, c_1, etc. so that $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Hint: first find the series for $f(x) = e^{-x}$

   Solution: Since $e^x = \Sigma_{n=0}^{\infty} x^n$, it follows that $e^{-x} = \Sigma_{n=0}^{\infty} (-x)^n$. Therefore $f(x) = (9x^2)e^{-x} = 9x^2 \Sigma_{n=0}^{\infty} (-x)^n$. The first few values of c_n are $c_0 = 0, c_1 = 0, c_2 = 9, c_3 = -9$ and $c_4 = 9/2$.

9. (10 points) Find the Maclaurin series of the function $f(x) = (7x^2)\sin(x)$. That is, find c_0, c_1, etc. so that $f(x) = \sum_{n=0}^{\infty} c_n x^n$.

   Solution: Just take the series for $\sin x$ and multiply it by $7x^2$. Thus, $(7x^2)\sin x = 7x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Thus, $c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 7/3! = 7/6, c_4 = 0$, etc.

10. (20 points) Suppose $\sum_{n=0}^{\infty} c_n (x-2)^n$ converges at $x = 3$.

    (a) Must it converge for $x = 0$?

    Solution: The value $x = 3$ is 1 unit from the center $a = 2$ so $R \geq 1$. This is not enough to guarantee convergence at 0.

    (b) Must it converge for $x = 1$?

    Solution: Since it could be the case that $R = 1$ and we could have divergence at the left endpoint, again there is no guarantee.

    (c) Must it converge for $x = 1.2$?

    Solution: The series must converge here because $|1.2 - 2| = 0.8 < 1$.

    (d) What can be said about the radius of convergence R?

    Solution: Only that $R \geq 1$.

(e) What can be said about the convergence of $\sum_{n=0}^{\infty} nc_n(1.5-2)^{n-1}$? How is this series related to the one above?

Solution: This is the series of $f'(x)$ which is known to converge for all $1 < x < 3$, because the radius of convergence of f and f' are the same.

11. (20 points) Find the Taylor polynomial $T_5(x)$ for $f(x) = \sin(2x)$ at $x = \pi$.

Solution: Build the table as shown

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin(2x)$	$\sin(2\pi) = 0$
1	$2\cos(2x)$	$2\cos(2\pi) = 2$
2	$-4\sin(2x)$	$-4\sin(2\pi) = 0$
3	$-8\cos(2x)$	$-8\cos(2\pi) = -8$
4	$16\sin(2x)$	$16\sin(2\pi) = 0$
5	$32\cos(2x)$	$32\cos(2\pi) = 32$

Therefore $T_5(x) = 2(x-\pi) - 8(x-\pi)^3/3! + 32(x-\pi)^5/5!$.

b. Find an upper bound for $|R_5(x)|$ on the interval $[\pi/2, 3\pi/2]$.

Solution: The sixth derivative of f is bounded by 64 over the interval in question, so $|R_5(x)| \leq 64(\pi/2)^6/6! = \pi^6/720 \approx 1.33$. There are much better methods, however, including the alternating series approximation. Points awarded based on students comments and work shown.

c. Find the radius of convergence of the Taylor series.

Solution: The factorial in the denominator guarantees that the limit of the ration in the ratio test is 0, so $R = \infty$.