Math 1165 **Discrete Math** **Final Exam**

December 15, 2005

Your name ______________________________

It is important that you **show your work**. The total value of this test is 260 points.

1. (15 points) For each sequence of integers below, determine whether the sequence is the degree sequence for a (simple) graph. If it is not, tell why not, and if it is, find a graph that has such a degree sequence.

   (a) 7 6 5 4 1 1 1 1 1

   (b) 7 6 5 4 2 2 2 2 1 1

   (c) 10 6 5 4 2 1 1 1 1 1

2. (15 points) Euler graphs and Hamilton graphs. Recall that a graph is called Eulerian if there is a circuit that used each edge exactly once. Its called Hamiltonian if there is a cycle (ie, a circuit that does not revisit any vertex) that contains all the vertices.

   (a) Give an example of a graph which is Eulerian but not Hamiltonian.

   (b) Give an example of a graph which is Hamiltonian but not Eulerian.

3. (40 points) Let $S = \{3, 4, 5, 6, 7, 8, 9\}$. Let D denote the set of all four-digit numbers that can be built using the elements of S as digits and allowing repetition of digits.

   (a) Find the number of four element subsets of S

   (b) How many subsets of S have at least three elements?

   (c) How many subsets of S have an odd number of members?

   (d) What is $|D|$? In other words, how many four-digit numbers can be written using only the digits 3,4,5,6,7,8, and 9?

   (e) How many elements of D have four different digits?

   (f) How many elements of D have exactly three different digits?

   (g) How many even numbers belong to D?

4. (20 points) Solve the Instant Insanity game shown below. In the nets at the bottom, list the colors which show in the positions determined by your solution.

	B		
R	B	W	G
	B		

CUBE 1

	R		
G	R	W	B
	W		

CUBE 2

	B		
B	R	G	G
	R		

CUBE 3

	B		
B	W	R	W
	W		

CUBE 4

	X		
	X		

CUBE 1

	X		
	X		

CUBE 2

	X		
	X		

CUBE 3

	X		
	X		

CUBE 4

5. (15 points) Consider the equation

$$x + y + z = 11,$$

where x, y, and z are integers. Recall that a solution is an ordered triple (x, y, z). Thus $(2, 3, 6) \neq (3, 2, 6)$.

   (a) How many solutions are there if x, y, and z are required to be positive?

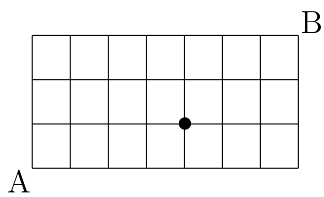
   (b) How many solutions are there if x, y, and z are required to be nonnegative?

   (c) How many solutions are there if x, y, and z are required to be nonnegative and $x \geq 2$?

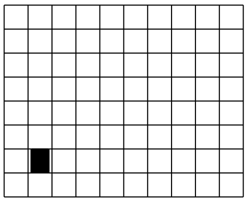
6. (15 points) Imagine that the 3×7 grid of squares below represents the streets of a part of the city where you live. You must walk 10 blocks to get from the lower left corner at A to the upper right corner at B.

   (a) How many different 10-block walks are there?

   (b) How many 10 block walks avoid the terrible corner marked with the bullet?



7. (15 points) An 8×10 grid of squares with one shaded square is given.



   (a) How many different squares are bounded by the gridlines?

   (b) How many different rectangles are bounded by the gridlines?

   (c) How many different rectangles bounded by the gridlines contain the shaded square?

8. (10 points) How many elements are in the union of four sets if each of the sets has 100 elements, each pair of sets share 50 elements, each triple of sets shares 25 elements and there are 5 elements in all four sets?

9. (15 points) Draw the 12-vertex knight's graph $GK(4,3)$ so that no two edges cross each other. This is the graph we used to solve the 4×3 knight interchange problem.

Part b. Is the graph $GK(4,3)$ Hamiltonian?

Part c. Is the graph $GK(4,3)$ Eulerian?

10. (15 points) Prove that

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$$

for $n = 0, 1, 2, \ldots$.

11. (15 points) Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 219 and 177. In other words, find integers x and y such that $gcd(219, 177) = 219x + 177y$.

12. (15 points) Find the base 4 and base -4 representation of each of the numbers below.

   (a) 217

   (b) 36.75

13. (20 points) Consider the following game. Two positive integers a and b are written on a board. The first player Al subtracts the smaller from the larger and writes that difference on the board. The second player Betty selects two numbers on the board and writes down the difference between the larger and the smaller. The winner is the last player to write down a new number.

    (a) Play this game when $a = 105$ and $b = 119$. What numbers are written on the board at the end?

    (b) Would you rather be Al or Betty? Why?

14. (20 points)

    (a) Prove that the union of two symmetric relations on the set A is also symmetric.

    (b) Prove that the union of two transitive relations need not be transitive.

15. (15 points) Let $S = \{1, 2, 3, 4\}$.

    (a) Find a relation on S with 6 ordered pairs that is symmetric and not transitive.

    (b) Find a relation on S with 5 ordered pairs that is antisymmetric and not transitive.

    (c) Find a relation on S with 7 ordered pairs that is transitive and not reflexive.