Throughout we use both the notations $\binom{n}{r}$ and C_r^n for the number $\frac{n!}{(n-r)!r!}$.

1. Ten points are distributed around a circle. How many triangles have all three of their vertices in this 10-element set?

   Solution: Each set of three points determines a triangle. The sampling takes place without replacement, and order does not matter. Therefore there are $\binom{10}{3} = 120$ triangles.

2. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set. Let D denote the set of all four-digit numbers that can be built using the elements of S as digits and allowing repetition of digits.

   (a) Find the number of four element subsets of S

   Solution: $C_4^{10} = 210$ subsets with four elements.

   (b) What is $|D|$? In other words, how many four-digit numbers are there?

   Solution: This is sampling with repetition and order matters. Therefore there are $E_4^{10} = 0.9(10^4) = 9000$ since the leftmost digit must not be 0.

   (c) How many elements of D have four different digits?

   Solution: This is sampling without repetition and order matters. Therefore there are $P_4^{10} = 10!/(10-4)! = 5040$. But this counts the ones that begin with a 0. Taking nine-tenths of 5040 give the answer: 4536.

   (d) How many elements of D have exactly three different digits?

   Solution: There must be two of one digit and one of two others. So, pick the duplicated digit in one of 10 ways, allowing zero, then pick the other two digits in $\binom{9}{2} = 36$ ways. Then select two locations for the duplicated digit $\binom{4}{2} = 6$ ways, and finally select one of the two orders for the other two digits: $10 \cdot 36 \cdot 6 \cdot 2 = 4320$. But this counts those that start with 0. Take 0.9 of 4320 to compensate. Thus, $.9 \cdot 4320 = 3888$. While we're at it, we ought to just count the ones with two distinct digits and those with one digit so that we can be sure we've counted correctly. There are $0.9 \cdot C_2^{10} \cdot [4 + 6 + 4] = 9 \cdot 9 \cdot 7 = 567$. There are 9 numbers that use just one digit, so we have a total of $4536 + 3888 + 567 + 9 = 9000$ four-digit numbers.

   (e) How many even numbers belong to D?

   Solution: Exactly half the 9000 members of D are even numbers, 4500.

3. The following problems are related.

(a) What is the value of $\frac{7!}{(7-3)!3!}$?

Solution: The value of $\frac{7!}{(7-3)!3!}$ is $\frac{7!}{(7-3)!3!} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$.

(b) How many 3-element subsets does the set $\{A, B, C, D, E, F, G\}$ have?

Solution: The number of three-element subsets of a seven element set is the same as the number of ways to *select* three items from a set of seven where we do not distinguish two samples iff the order differs, and we sample without replacement. The typical notation for this number is ${}^7C_3$, or C_3^7, or $\binom{7}{3}$. There is a formula for C_r^n. It is $C_r^n = \frac{n!}{(n-r)!r!}$. In our special case, this is $\frac{7!}{(7-3)!3!} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$.

(c) How many solutions are there to

$$x + y + u + v = 4$$

where x, y, u, and v are nonnegative integers. For example, $(2, 1, 0, 1)$ is such a solution.

Solution: Consider the seven numbered blanks

$$\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}$$

Let us agree to select four of these blanks in which to put 1's, and to put dots in the other three places. For example, $11 \cdot 1 \cdot \cdot 1$. Now this string is a coding for the solution $2, 1, 0, 1)$ of the problem. How many solutions are there? The answer is that there is one solution for each way of selecting the positions in which to put the 1's, or equivalently, one solution for each way to select the three dots, which we know from the last problem is 35.

(d) How many solutions does

$$x + y + u + v = 8$$

have subject to the condition that each of the variables is a positive integer?

Solution: For each solution (a, b, c, d) of problem 2, $(a+1, b+1, c+1, d+1)$ is a solution to this problem. Thus the answer is 35. Alternatively, consider the seven gaps determined by a string of eight 1's:

1_1_1_1_1_1_1_1_.

Select three of these seven positions in which to insert dots. For example

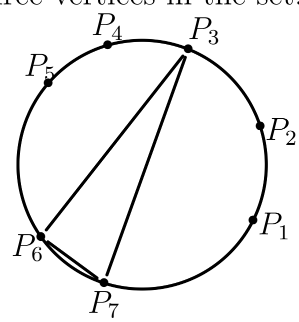
1·1_1_1_1·1_1·1_1

corresponds to the solution $(1, 3, 2, 2)$.

(e) How many ways can a 3-person committee be selected from a 7-member club?

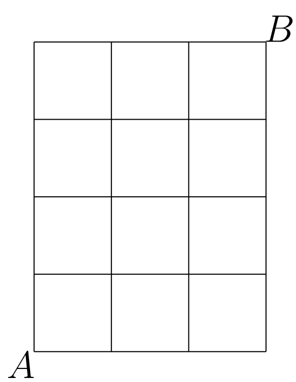
Solution: Name the members of the club A, B, C, D, E, F, G, so selecting the three-person committee is the same as the first problem of selecting a three-member subset of a seven member set.

(f) Let $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ be seven points distributed around a circle. How many triangles have all three vertices in the set.



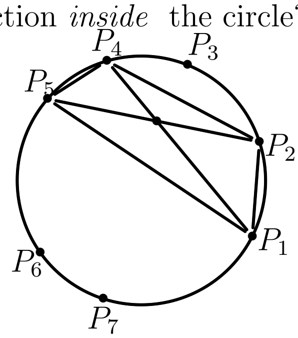
Solution: Every three points determines a triangle.

(g) How many paths of length 7 are there from A to B in the grid below?



Solution: 35. This is just the Polya Block-walking model again.

(h) Seven points are distributed around a circle. All pairs of them are joined by a secant line. What is the largest possible number of points of intersection *inside* the circle?



Solution: Each intersection point is determined by four of the seven points on the circle. The point I shown below is an example of one such point.

(i) What is the coefficient of x^3 in the expanded form of $(x+1)^7$?

Solution: The entries of the seventh row of Pascal's triangle are exactly the coefficients we need, and the coefficient of x^3 is the number 35.

(j) What is the third entry of the seventh row of Pascal's triangle?

Solution: The seventh row is 1 7 21 35 35 21 7 1, so again the answer is 35.

(k) How many numbers can be expressed as a sum of four distinct members of the set $\{1, 2, 4, 8, 16, 32, 64\}$?

Solution: Notice that each member of the set is larger than the sum of all the smaller members of the set. Therefore all the sums are unique. There are $\binom{7}{4} = 35$ such sums.