Throughout we use both the notations $\binom{n}{r}$ and C_r^n for the number $\frac{n!}{(n-r)!r!}$.

1. A *falling* number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example 96521 is a falling number but 89642 is not. How many n-digit falling numbers are there, for $n = 1, 2, 3, 4, 5, 6, 7, 8,$ and 9? What is the total number of falling numbers of all sizes?

   Solution: Each set of k digits can be arranged in exactly one way to form a falling number. Thus, there are just as many k-digit falling numbers as there are k-member subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. That is, $\binom{10}{1}\binom{10}{2}$, $\binom{10}{3}$, $\binom{10}{4}$, $\binom{10}{5}$, ..., $\binom{10}{10}$. Adding all these numbers together gives $2^{10} - 1 = 1023$, since we don't count numbers with zero digits.

2. Cyprian writes down the middle number in each of the $\binom{9}{5} = 126$ five-element subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then he adds all these numbers together. What sum does he get?

   Solution: He gets $3 \cdot \binom{6}{2} + 4 \cdot \binom{3}{2} \cdot \binom{5}{2} + 5 \cdot \binom{4}{2} \cdot \binom{4}{2} + 6 \cdot \binom{5}{2} \cdot \binom{3}{2} + 7 \cdot \binom{6}{2} \cdot \binom{2}{2} = 45 + 120 + 180 + 180 + 105 = 630$. Alternatively, there is a 1-1 correspondence between the five-element subsets with middle value k and those with middle value $10-k$, so the average value of the middle number is 5. Since there are $\binom{9}{5} = 126$ five-element subsets, the sum Cyprian gets is $5 \cdot 126 = 630$. An example of the 1-1 correspondence is given below: $\{2, 3, 6, 7, 8\} \Leftrightarrow \{2, 3, 4, 7, 8\} = \{10-8, 10-7, 10-6, 10-3, 10-2\}$.

3. Counting sums of subset members.

   (a) How many number can be expressed as a sum of two or more distinct elements of the set $\{1, -3, 9, -27, 81, -243\}$?

   Solution: An equivalent question is How many numbers have base -3 representations that have at most six digits all of which are 1's and 0's. The answer to the second question is 7 more than the answer to the first one however, because of the requirement that we must add two or more of the powers of -3. There are $2^6 = 64$ such six-digit numbers, so the answer to our question is $64 - 7 = 57$.

   (b) How many numbers can be expressed as a sum of two or more distinct members of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

   Solution: The sums are not distinct. However the numbers we can get are consecutive integers. The smallest is $1 + 2 = 3$ and the largest is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, so there are $45 - 2 = 43$ such numbers.

(c) How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

Solution: Each member of $\{17, 21, 25, 29, 33, 37, 41\}$ is one more than a multiple of four. Therefore, any sum of four of them is a multiple of 4. The smallest such number is $17 + 21 + 25 + 29 = 92$ and the largest is $29 + 33 + 37 + 41 = 140$ and all the multiples of 4 between them are obtainable. There are $\frac{140-92}{4} + 1 = \frac{48}{4} + 1 = 13$ such numbers.
Alternatively, we transform the problem into a simpler one. Because $17 = 4 \cdot 4 + 1, 21 = 4 \cdot 5 + 1, \ldots, 41 = 4 \cdot 10 + 1$, it makes sense to set up a correspondence between the set of sums R and the set of numbers generated by $\{4, 5, 6, 7, 8, 9, 10\}$ OR between R and those numbers generated by $\{-3, -2, -1, 0, 1, 2, 3\}$. The set R in this case is just $\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$, which has 13 members.

(d) How many numbers can be expressed as a sum of two or more distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

Solution: We must look at cases. Solution to be supplied later.

4. How many of the first 242 positive integers are expressible as a sum of three or fewer members of the set $\{3^0, 3^1, 3^2, 3^3, 3^4\}$ if we are allowed to use the same power more than once. For example, $5 = 3 + 1 + 1$ can be represented, but 8 cannot. Hint: think about the ternary representations.

Solution: The number of powers of 3 used is just the sum of the ternary digits. We are looking for the number of integers whose ternary representations have a sum of digits of 3 or less. We consider three types, those with sum of digits 1, 2 and 3. There are just 5 with sum of digits 1: $10000, 1000, 100, 10, 1$. There are $\binom{5}{2} + \binom{5}{1} = 10 + 5 = 15$ numbers with sum of digits 2. There are two types, those with two 1's and those with one 2. There are $\binom{5}{3} + P_2^5 = 10 + 20 = 30$ of the third type for a total of $5 + 15 + 30 = 50$ numbers altogether.

5. How many integers can be expressed as a sum of two or more different members of the set $\{0, 1, 2, 4, 8, 16, 32\}$?

Solution: The 0 in the set means that we should count the individual elements among the sums. Every number from 1 and 63 has a unique representation (its binary representation) as a sum of powers of 2. The powers of two themselves can be represented as a sum because of the 0 in the set. Thus the answer is 63.

6. John has 2 pennies, 3 nickels, 2 dimes, 3 quarters, and 8 dollars. For how many different amounts can John make an exact purchase (with no change required)?

   Solution: We'll count achievable amounts less than \$1, and multiply by 9, then add in the 9 values $9.00, 9.01, 9.02, 9.05, 9.06, 9.07, 9.10, 9.11, 9.12$. They are $\{0, 1, 2, 5, 6, 7, 10, 11, 12, \ldots\}$, exactly three of every five consecutive values. So, counting 0, there are $\frac{3}{5}(100) = 60$ such values. Hence there are $9 \times 60 - 1 = 539$, since we don't count the value 0. Now adding in the nine uncounted values, we get $539 + 9 = 548$.

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