November 22, 2004 **Name** ______________________________

The total number of points available is 195. Throughout this test, **show your work.** For most questions, the answer along without any supporting mathematics will not be worth any points.

1. (40 points) Find each of the following derivatives.

   (a) If $f(x) = e^{x^2}$, then $f'(x) =$
   Solution: $2xe^{x^2}$.

   (b) If $g(x) = \ln(x^2 - 2x + 2)$, then $g'(x) =$
   Solution: $\frac{2x-2}{x^2-2x+2}$.

   (c) If $y = xe^{x^2-x}$, then $\frac{dy}{dx} =$
   Solution: By the product rule, $y' = e^{x^2-x} + x(2x-1)e^{x^2-x} = e^{x^2-x}(2x^2 - x + 1)$.

   (d) If $h(x) = \dfrac{\ln x}{x}$, then $h'(x) =$
   Solution: By the quotient rule, $h' = \frac{1-\ln x}{x^2}$.

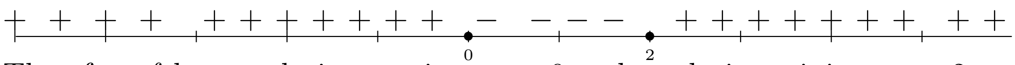
   (e) If $y = (\ln e^{2x-1})^2$, then $\frac{dy}{dx} =$
   Solution: Massage the function to get $y = (2x-1)^2$, so the derivative is $y' = 2(2x-1) \cdot 2 = 8x - 4$.

2. (20 points) Consider the function $f(x) = e^{x^3-3x^2}$.

   (a) Compute $f'(x)$.
   Solution: $f'(x) = (3x^2 - 6x)e^{x^3-3x^2}$.

   (b) Find the critical points of f.
   Solution: $(3x^2 - 6x) = 3x(x - 2) = 0$ at $x = 0$ and $x = 2$.

   (c) Find the relative max and min of f.
   Solution: The sign chart for f' is



   Therefore f has a relative maximum at 0 and a relative minimum at 2.

   (d) What is the maximum value of f over the interval $[-2, 4]$?
   Solution: Examine the function at the points $-2, 0, 2$ and 4. You find that $f(-2) = e^{-20}, f(0) = 1, f(2) = e^{-4}$ and $f(4) = e^{16}$, the last of which is enormously greater than all the others.

3. (10 points) Find an equation for the line tangent to the graph of $y = \ln x^2$ at the point $(1, 0)$.

   Solution: The derivative of the function is $y' = 2/x$ so the slope of the line at $x = 1$ is $2/1 = 2$ and the line is $y - 0 = 2(x - 1)$, That is $y = 2x - 2$.

4. (10 points) Find an equation for the line tangent to the graph of $y = e^{4x-2}$ at the point $(1/2, 1)$.

   Solution: $y' = 4(e^{4x-2}$ so $m = 4e^{4\cdot\frac{1}{2}-2} = 4$. Thus the line is $y - 1 = 4(x - 1/2)$.

5. (20 points) A skull from an archeological dig has one-tenth the amount of Carbon-14 it had when the specimen was alive.

   (a) Recall that the half-life of Carbon-14 is 5770 years. Find the decay constant k.

   Solution: We must solve the equation $Q(5770) = Q_0/2$ for k, where $Q(t) = Q_0e^{-kt}$. The equation leads to $e^{-5770k} = 1/2$ which means that $k \approx 1.201 \times 10^{-4}$.

   (b) What is the age of the specimen?

   Solution: To solve the equation $Q_0e^{-kt} = Q_0/10$, divide both sides by Q_0 to get $e^{-kt} = 0.1$. This happens for $t = 19167$ years.

6. (20 points) How long does it take an investment of \$1000 at an annual rate of 6% to triple in value if compounding takes place

   (a) takes place quarterly? Round your answer to the nearest tenth of a year.

   Solution: We need to solve the equation $3000 = 1000\left(1 + \frac{0.06}{4}\right)^{4t}$ for t. Using logs, we get $t = \frac{\ln 3}{4\ln 1.015} \approx 18.447 \approx 18.4$ years.

   (b) takes place continuously? Round your answer to the nearest tenth of a year.

   Solution: We need to solve the equation $3000 = 1000e^{0.06t}$ for t. Using logs, we get $t = \frac{\ln 3}{0.06} \approx 18.31 \approx 18.3$ years.

7. (25 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where T is the air temperature, $72°F$, A and k are constants, and t is expressed in minutes.

(a) What is the value of A?

Solution: Note that $F(0) = 72 + A \cdot 1 = 212$ so $A = 140$.

(b) Suppose that after exactly 20 minutes, the temperature of the coffee is $186.6°F$. What is the value of k?

Solution: Solve $F(t) = 186.6 = 72 + 140e^{-k(20)}$ for k to get $k = 0.01000973$.

(c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

Solution: Solve the equation $80 = 72 + 140e^{-0.020019t}$ for t to get first $e^{-0.010009t} = 8/140 = 0.05714$, and taking logs of both sides yields $t = 285.9$ minutes.

(d) Find the rate at which the object is cooling after $t = 20$ minutes.

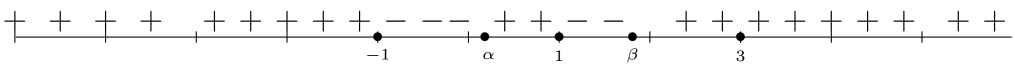
Solution: To find $F'(t)$ recall the way we differentiate exponential functions. $F'(t) = 140(-k)e^{-kt}$, so $F'(20) = 140(-k)e^{-20k} = -1.147$ degrees per minute.

8. (40 points) Consider the function $g(x) = (x-3)^3(x-1)^2(x+1)^4$.

   (a) Compute $\frac{dy}{dx}$ using logarithmic differentiation.
   Solution: If $y = (x-3)^3(x-1)^2(x+1)^4$, then $\ln y = 3\ln(x-3)+2\ln(x-1)+4\ln(x+1)$. It follows that $y'/y = \left(\frac{3}{x-3}+\frac{2}{x-1}+\frac{4}{x+1}\right)$. Therefore, we can write $y' = y\left(\frac{3}{x-3}+\frac{2}{x-1}+\frac{4}{x+1}\right)$.

   (b) Find the critical points of g.
   Solution: Continuing from above and doing a lot of polynomial algebra, $y' = (x-3)^2(x-1)(x+1)^3(9x^2-18x+3)$. Of course the zeros of g' are $x = 3, x = 1$ and $x = -1$. But there are two others which we get from using the quadratic formula on $9x^2-18x+3$. These numbers are roughly $\alpha = 0.0183$ and $\beta = 1.816$.

   (c) Construct the sign chart for $g'(x)$.
   Solution: Now the sign chart for g' yields:



   (d) (added after the test) Use the sign chart to find the relative extrema of g.
   Solution: We can read off the relative extrema from the chart. The relative mins occur at $x = \alpha$ and $x = \beta$, while the relative maxes occur at -1 and 1. Note that g' does not change sign at 3.

9. (30 points) Recall that up to this chapter in the course, there are just two types of functions whose (implied) domains are restricted. For example $y = 1/x$ has domain $(-\infty, 0) \cup (0, \infty)$ and $y = \sqrt{x}$ has domain $[0, \infty)$. We now have a third such function type. The domain of $y = \ln x$ is $(0, \infty)$.

   (a) What is the domain of $f(x) = \ln((x-1)(x+2))$?
   Solution: The domain is just $D = (-\infty, -2) \cup (1, \infty)$.

   (b) Compute $f'(x)$ and construct its sign chart. Note that $f'(x)$ appears to have a larger domain than f. Of course that is impossible.
   Solution: $f'(x) = \frac{2x+1}{(x-1)(x+2)}$ over D. So the sign chart looks like:

   (c) Over what intervals is f increasing?
   Solution: It is now easy to read from the sign chart for f' that f is increasing for $x > 1$

   (d) Over what intervals is f concave upwards?
   Solution: First use the quotient rule to compute f''.

$$\begin{aligned} f''(x) &= \frac{2(x^2+x-2)-(2x+1)(2x+1)}{(x^2+x-2)^2} \\ &= \frac{(2x^2+2x-4)-(4x^2+4x+1)}{(x^2+x-2)^2} \\ &= \frac{-2x^2-2x-5}{(x^2+x-2)^2} < 0 \end{aligned}$$

   for all $x \in D$. Thus, f is not concave up anywhere.