April 26, 2003 **Name** ____________________

In the first five problems, each part counts 5 points and the final two problems count as marked. There are 120 points available on this test.

Multiple choice section. Circle the correct choice. You do not need to show your work on these problems.

1. Let $f(x) = e^{2x-1}$. What is $f'(0)$?

   (A) 1 **(B)** 2 **(C)** $2e^{-1}$ **(D)** e^1 **(E)** $2e^1$

   Solution: $2e^{-1}$

2. Let $f(x) = \ln(x^4)$. What is $f'(e^2)$?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** $4e^{-2}$ **(E)** $4e^2$

   Solution: $4e^{-2}$

3. For how many values of x is it true that

   $$\ln[(x^2-1)(x^2-4)+e] = 1?$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: The 4 values of x are ± 1 and ± 2.

4. Population Growth. The population of a town increases according to the model

   $$P(t) = 2200e^{0.04t}$$

   where t is measured in years with $t = 0$ corresponding to 1990. Use the model to approximate the population in 1995.

   (A) 2599 **(B)** 2608 **(C)** 2655 **(D)** 2679 **(E)** 2687

   Solution: 2687

5. A total of \$10,000 is invested at an annual rate of 10%. Find the balance (to the nearest dollar) after 5 years if it is compounded quarterly.

   (A) $\$16,386$ **(B)** $\$16,683$ **(C)** $\$16,720$ **(D)** $\$16,818$ **(E)** $\$16,988$

   Solution: Note that $(1+.1/4)^{20} \approx 1.6386$ so the amount in the account after five years is $\$16,386$.

On all the following questions, **show your work.**

6. (15 points)

   (a) Find the derivative of the function $g(x) = (1 + e^{2x})^{3/2}$.
   Solution: $g'(x) = 3e^{2x} \cdot \sqrt{1 + e^{2x}}$.

   (b) Find the derivative of the function $f(x) = \ln(1 + e^{3x})$.
   Solution: $f'(x) = 3e^{3x} \div (1 + e^{3x})$.

   (c) Solve the equation $5e^{x^2+1} + 3 = 2003$?
   Solution: First subtract 3 from both sides, then divide by 5 to get $e^{x^2+1} = 400$. Take natural logs of both sides to get $x^2 + 1 = \ln(400) \approx 5.9914$. Finally, subtract 1 from both sides and take the square root to get $x \approx \sqrt{4.9914} \approx 2.234$.

7. (20 points) Suppose that \$300 is deposited into an account with an annual percentage rate of 7%. What is the balance in the account after 4 years, assuming that compounding takes place

   (a) quarterly? Round your answer to the nearest penny.
   Solution: $300(1 + \frac{0.07}{4})^{4 \cdot 4} \approx 300 \cdot 1.3199\ldots \approx \$395.98$.

   (b) continuously? Again, round your answer to the nearest penny.
   Solution: Use the shampoo formula, $A = Pe^{rt}$ to get $A = 300e^{0.07 \cdot 4} \approx 300 \cdot 1.3231 \approx \$396.94$.

8. (20 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where T is the air temperature, $72°F$, A and k are constants, and t is expressed in minutes.

(a) What is the value of A?

Solution: Note that $F(0) = 72 + A \cdot 1 = 212$ so $A = 140$.

(b) Suppose that after exactly 10 minutes, the temperature of the coffee is $186.6°F$. What is the value of k?

Solution: Solve $F(t) = 186.6 = 72+140e^{-k(10)}$ for k to get $k = 0.020019$.

(c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

Solution: Solve the equation $80 = 72 + 140e^{-0.020019t}$ for t to get first $e^{-0.020019t} = 8/140 = 0.05714$, and taking logs of both sides yields $t = 142.97$ minutes.

9. (20 points) A radioactive substance has a half-life of 66 years.

   (a) Use the exponential decay model to write an equation which shows that after 66 years, a sample with 200 grams of radioactivity has only 100 grams left.

   Solution: The amount of radioactive substance A left is given by

   $$A(t) = Ie^{-kt}$$

   where k is a positive constant, and I is the initial amount. Then $A(66) = Ie^{-66k} = 100$grams.

   (b) Use the fact that there are **initially** 200 grams of radioactivity to solve for one of the functions parameters.

   Solution: Because the original amount satisfies $A(0) = Ie^{-k\cdot 0} = Ie^0 = I$, it follows that $I = 200$.

   (c) Use the information in (a) and (b) to solve for the constant k in the function.

   Solution: Note that $100 = 200e^{-66k}$ is equivalent to saying $e^{-66k} = .5$. Solve this by taking the natural log of both sides: $k = \ln(.5)/-66 = 0.0105$.

   (d) How many years must elapse before the amount of radioactivity is down to 25 grams.

   Solution: $25 = 200e^{-0.0105t}$ which means that $e^{-0.0105t} = 0.125$. Take the natural log of both sides to get $t = \frac{\ln(1/8)}{-0.0105} = 198$ years.

10. (20 points) Consider the logistic growth function

$$Q(t) = \frac{A}{1 + Be^{-kt}},$$

where A, B, and k are positive constants.

(a) Find $Q'(t)$.
Solution:

$$\begin{aligned} Q'(t) &= \frac{0 - A(-kBe^{-kt})}{(1 + Be^{-kt})^2} \\ &= \frac{kABe^{-kt}}{(1 + Be^{-kt})^2} \end{aligned}$$

(b) Use this information to show that $Q(t)$ is increasing in the interval $(0, \infty)$.
Solution: Both numerator and denominator of $Q'(t)$ are positive, so $Q'(t) > 0$ for $t > 0$. Thus $Q(t)$ is an increasing function.