December 2, 1999
Multiple choice section. Circle the correct choice. You do not need to show your work on these problems.

1. Let $f(x) = e^{2x+1}$. What is $f'(0)$?

   (A) 0 **(B)** e **(C)** $\boxed{2e}$ **(D)** e^2 **(E)** $2e^2$

1b. Let $f(x) = e^{2x-1}$. What is $f'(0)$?

   (A) 0 **(B)** $1/e$ **(C)** $\boxed{2/e}$ **(D)** $1/e^2$ **(E)** $2/e^2$

2. An amount P is invested at 6.75% compounded monthly. To the nearest hundredth of a year, how long does it take before the investment is worth $3P$?

   (A) 16.22 **(B)** $\boxed{16.32}$ **(C)** 16.42 **(D)** 16.52 **(E)** 195.88

3. Let $h(x) = \ln(2x + 3)$. What is $h'(-1)$?

   (A) -1 **(B)** $-1/2$ **(C)** 1 **(D)** $\boxed{2}$ **(E)** it does not exist

   $\boxed{h'(x) = \frac{2}{2x+3} \text{ so } h'(-1) = 2.}$

4. What is the slope of the line tangent to the graph of $h(x) = 2e^x + e^{-2x}$ at the point $\ln 2, 4.25)$?

   (A) -2 **(B)** -1.25 **(C)** 0 **(D)** 2.25 **(E)** $\boxed{3.5}$

5. What is the slope of the line tangent to the graph of $f(x) = \ln x$ at the point $(2, \ln 2)$?

   (A) 0 **(B)** $\boxed{1/2}$ **(C)** 1 **(D)** 2 **(E)** e

5b. Which of the following is a stationary point of the function $h(x) = 2e^x + e^{-2x}$?

   (A) -2 **(B)** -1 **(C)** $\boxed{0}$ **(D)** 1 **(E)** the function h has no stationary points

6. A total of \$10,000 is invested at an annual rate of 8%. Find the balance after 5 years if it is compounded continuously, to the nearest dollar.

   (A) \$14,859 **(B)** \$14,903 **(C)** $\boxed{\$14,918}$ **(D)** \$14,928 **(E)** \$14,938

6b. A total of \$10,000 is invested at an annual rate of 7%. Find the balance after 5 years if it is compounded continuously, to the nearest dollar.

(A) $\boxed{\$14,190}$ **(B)** $\$14,197$ **(C)** $\$14,207$ **(D)** $\$14,212$ **(E)** $\$14,219$

7. $\lim_{x\to 0} 4 - 2e^{-3x} =$

(A) 0 **(B)** $\boxed{2}$ **(C)** 4 **(D)** 10 **(E)** This limit does not exist.

7b. $\lim_{t\to\infty} \dfrac{3}{2+e^{-3t}} =$

(A) 0 **(B)** $\boxed{3/2}$ **(C)** 2 **(D)** 3 **(E)** This limit does not exist.

On all the following questions, **show your work.**
The next four problems count 5 points each. Compute the following derivatives.

8. $\dfrac{d}{dx} xe^{4x}$

By the product rule, $\frac{d}{dx}xe^{4x} = 1 \cdot e^{4x} + 4x \cdot e^{4x}$

9. $\dfrac{d}{dx} \ln(x-3)^3(x^2+1)^2$

First, rewrite the function to get $D = \dfrac{d}{dx}\ln(x-3)^3(x^2+1)^2 = \dfrac{d}{dx}(3\cdot\ln(x-3)+2\cdot(x^2+1))$, and then follow the rule for differentiating the natural log of a function: $D = \frac{3}{x-3} + 2\cdot\frac{2x}{x^2+1} = \frac{3}{x-3} + \frac{4x}{x^2+1}$.

10. $\dfrac{d}{dx} \ln(e^{x^3})$

Recall that $\ln(e^{f(x)}) = f(x)$, so the problem is simply to differentiate the function x^3. The desired answer is just $3x^2$.

removed $\dfrac{d}{dx} x \cdot \ln(x^2 + 4x + 4)$

$\boxed{\text{Notice that } x^2 + 4x + 4 = (x+2)^2 \text{ so the function can be written as } 2x \cdot \ln(x+2) \text{ whose derivative is just } 2 \cdot \ln(x+2) + \tfrac{1}{x+2} \cdot 2x.}$

On all the following questions, **show your work.**
The next three problems count 5 points each. Compute the following antiderivatives.

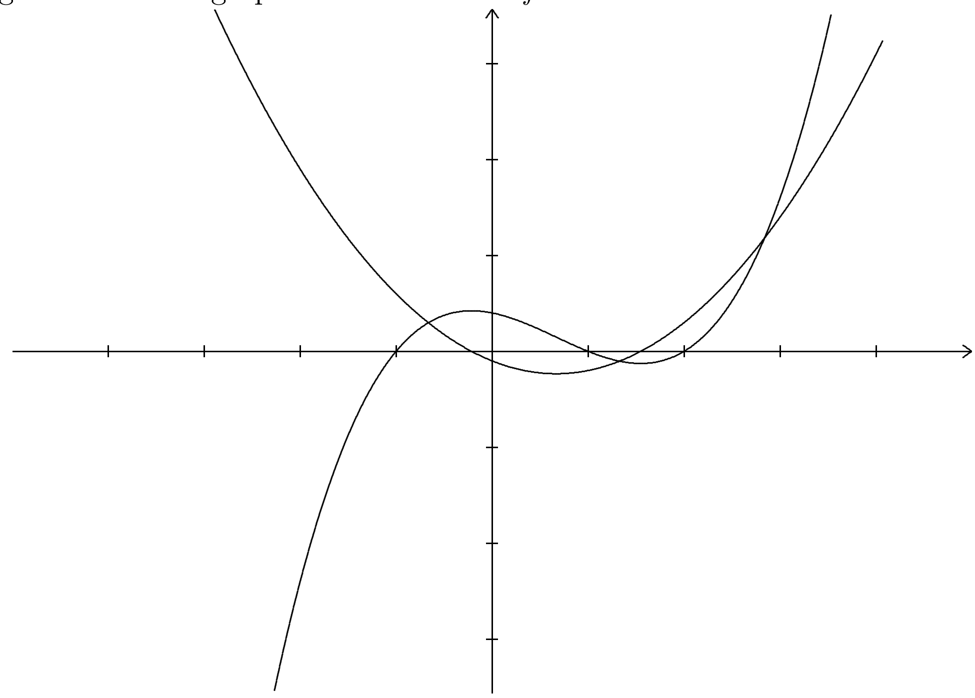
11. $\int 4x^2 + 2x - 1 dx$ $\boxed{= \frac{4}{3}x^3 + x^2 - x + c}$

11b. $\int 12x^2 + 4x - 1 dx$ $\boxed{= 4x^3 + 2x^2 - x + c}$

12. $\int 6x^{\frac{3}{2}} + x^{-\frac{1}{2}} dx$ $\boxed{= 6 \cdot 2/5 \cdot x^{5/2} + 2x^{1/2} + c}$

13. $\int \dfrac{3x^2 + 2x - 1}{x} dx$ $\boxed{= \int (3x + 2 - 1/x)\, dx = 3x^2/2 + 2x - \ln|x| + c}$

14. (10 points) On the grid below, the graph of a function f is drawn. On the same grid sketch the graph of its derivative f'.



15. (15 points) Let g be defined by $g(x) = 2x^4 - x^2 - 3$ on the interval $[-3, 3]$.

(a) Find the first and second derivatives of g.

Note that $g'(x) = 8x^3 - 2x$, and $g''(x) = 24x^2 - 2$.

(b) Find all the stationary points of g.

Solve $8x^3 - 2x = 0$ to get $x = 0$ and $x = \pm 1/2$.

(c) Find all values of x where g has relative extrema.

Note that $g''(x)$ is not zero at any of the three values in (b), so all are locations of extrema.

(d) Use the second derivative test to identify each (c) as the location of a relative max or a relative min.

Note that $g''(0) = -2 < 0$ so a maximum occurs at 0, and that $g''(\pm 1/2) = 4 > 0$ so minimums occur at these two values.

(e) Find a point of inflection of g.

Solve $24x^2 - 2 = 0$ to get $x = \pm\sqrt{1/12}$.

(f) Find the absolute maximum and absolute minimum of g. You must show why all the other candidates were rejected to get full credit.

Just evaluate g at the endpoints ± 3 and at the stationary points to find the absolute max and min: $g(\pm 3) = \pm 3^4 - \pm 3^2 - 3 = 150$ (note that g is an even function $g(-x) = g(x)$; and $g(\pm 1/2) = -3.0125$, while $g(0) = -3$. Thus the absolute max value of g is 150 and the absolute minimum value is -3.0125.

16. (20 points) A radioactive substance has a half-life of 33 years.

(a) Find the function $Q(t)$ that gives the units of radioactivity left as a function of time t in years.

$\boxed{Q(t) = Q_0 e^{-kt}.}$

(b) Suppose we begin with a sample of 200 grams of the substance. Write an equation using the function in (a) that describes the fact that after 33 years, exactly 100 grams of the substance is left.

$\boxed{Q(33) = 100 = 200e^{-k \cdot 33}.}$

(c) Use the information in (a) and (b) to find the unknown parameters of the function.

Solve $100 = 200e^{-k \cdot 33}$ for k to get $k = (\ln 2)/33$. Of course the value of Q_0 is 200.

(d) How many grams (to the nearest tenth of a gram) of the substance is left after 50 years?

Since $Q(t) = 200 \cdot e^{-(\ln 2)t/33}$, it follows that $Q(50) = 200 \cdot 2^{-\frac{5}{3}} \approx 62.996 \approx 63.0$.

(e) How many years must elapse before the amount of radioactivity is down to 25 grams.

You can get this without doing anything technical. Note that after 33 years, 100 grams are left. After another 33 years, 50 grams are left, and after yet another 33 years, 25 grams are left. Therefore it takes 99 years to decay to 25 grams.