April 13, 2011 **Name** ______________________________

The problems count as marked. The total number of points available is 159. Throughout this test, **show your work.** Use calculus to work the problems. Calculator solutions which circumvent the ideas and techniques of the course will typically be worth about one fourth credit.

1. (12 points) Find an equation for the line tangent to the graph of $f(x) = xe^{-2x+4}$ at the point $(2, f(2))$.

   Solution: Find f' first. $f'(x) = e^{-2x+4} - 2xe^{-2x+4}$. Then note that $f'(2) = 1 + 2(-2) \cdot 1 = -3$ and $f(2) = 2$, so the line is $y = -3x + 8$.

2. (12 points) Find an equation for the line tangent to the graph of $g(x) = (x + \ln(x))^2$ at the point $(1, 1)$.

   Solution: Find g' first. Then note that $g'(x) = 2(x+\ln(x))(1+\frac{1}{x})$, so $g'(1) = 4$ and, so the line is $y - 1 = 4(x - 1)$ or $y = 4x - 3$.

3. (25 points) Find a symbolic representation of a rational function $r(x)$ that has all the following properties:

   (a) i. It has exactly two zeros, $x = -1$ and $x = 2$.
   ii. It has two vertical asymptotes, $x = -2$ and $x = 3$.
   iii. It has $y = 3$ as a horizontal asymptote.

   Solution: There are a few ways to do this. The easiest is to make the numerator $3(x + 1)(x - 2)$ and the denominator $(x + 2)(x - 3)$.

   (b) Find the derivative of your function.

   Solution: Use the quotient rule. You get $r'(x) = \frac{3(2x-1(x^2+x-2)-3(2x+1)(x^2-x-2)}{(x^2+x-2)^2} = \frac{x^2+4}{(x^2+x-2)^2}$, which is positive wherever it is defined. Therefore r is increasing on all three intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$.

   (c) Use the information in part b) to find the intervals over which your function is increasing.

   Solution: This will change from one function to the next. For the proposed function, r is increasing on all three intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$.

4. (30 points) Consider the rational function r defined by

$$r(x) = \frac{(2x^2 + 3x - 14)(x^2 - 1)}{(x^2 + x - 6)(x + 1)^2}.$$

(a) Find the zeros of r.

Solution: After cancelling the common factors, we find the function is essentially

$$r(x) = \frac{(2x + 7)(x - 1)}{(x + 3)(x + 1)},$$

so the zeros are $x = -7/2$ and $x = 1$.

(b) Find the vertical asymptotes of r.

Solution: The zeros of the denominator are $x = -3$ and $x = -1$.

(c) Does the function have any horizontal asymptotes? If so, what are they?

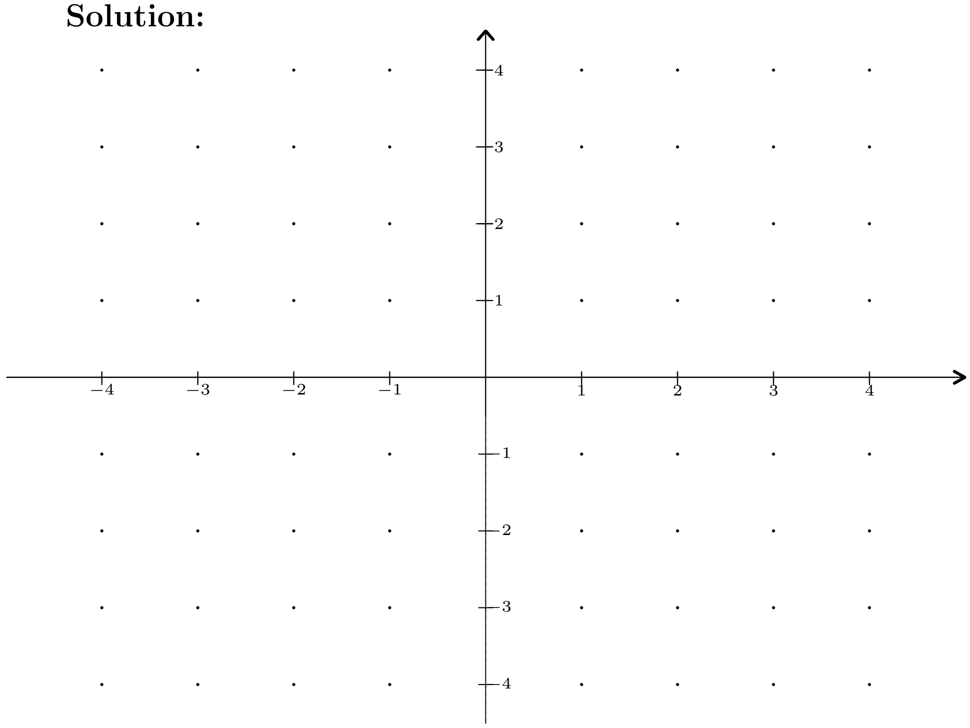
Solution: $y = 2$ is the horizontal asymptote.

(d) Build the sign chart for r.

Solution: The function is positive on $(-\infty, -7/2) \cup (-3, -1) \cup (1, \infty)$.

(e) Using precisely the information you found in the first three parts of the problem, sketch the graph of r.

Solution:



5. (20 points) Certain radioactive material decays in such a way that the mass remaining after t years is given by the function

$$m(t) = 165e^{-0.02t}$$

where $m(t)$ is measured in grams.

(a) Find the mass at time $t = 0$.
Solution: $m(0) = 165$ grams.

(b) How much of the mass remains after 15 years?
Solution: $m(15) = 122.23$ grams.

(c) What is the half-life of the material?
Solution: Solve the equation $e^{-0.02t} = \frac{82.5}{165} = \frac{1}{2}$ which leads to $-0.02t = \ln(0.5)$ for which the value of t is about 34.6 years.

(d) Find the rate of loss at $t = 1$ year.
Solution: Since $m'(t) = 165(-0.02)e^{-0.02t}$, it follows that $m'(1) \approx -3.234$ grams per year.

6. (15 points) Four identical $x \times x$ square corners are cut from a 12×18 inch rectangular piece of metal, and the sides are folded upward to build a box.

(a) What is the volume of the box that results when the corners cut are 1×1.
Solution: The volume is the product length $\times$ width $\times$ height $= 10 \times 16 \times 1 = 160$.

(b) Let $V(x)$ denote the volume of the box when the $x \times x$ corners are removed. Find $V(2)$ and $V(3)$.
Solution: Note that $V(x) = (12-2x)(18-2x)(x)$ and $V(2) = 8 \cdot 14 \cdot 2 = 224$ and $V(3) = 6 \cdot 12 \cdot 3 = 216$.

(c) What is the implied domain of V?
Solution: The domain of V is $[0, 6]$.

(d) Find $V'(x)$.
Solution: $V'(x) = \frac{d}{dx}(12x - 2x^2)(18-2x) = (12-4x)(18-2x) + (12x - 2x^2)(-2) = 12x^2 - 120x + 216$.

(e) Find the critical points of $V(x)$.
Solution: Factor out 12 from $V'(x)$ and use the quadratic formula to find the zeros of $x^2 - 10x + 18$. We get

$$x = \frac{10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 18}}{2} = \frac{10 \pm \sqrt{28}}{2} = 5 \pm \sqrt{7} \approx 2.35$$

when we take the negative sign. The other place where $V'(x) = 0$ is outside the interval $[0, 6]$. Its about 7.65.

(f) What value of x makes the value of V maximum? Estimate within 0.01 the maximum value of V.

Solution: $V(2.35) \approx 228.1621 \approx 228.16$.

7. (20 points) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $f(x) = 16 - x^2$. For example two of the vertices of the rectangle could be $(-3, 0)$ and $(3, 0)$, both on the x-axis. Then the other two vertices would be $(3, f(3)) = (3, 16 - (3)^2) = (3, 7)$ and $(-3, f(-3)) = (-3, 7)$. In this case the area of the rectangle is $A = 6 \cdot 7 = 42$.

   (a) Now suppose we use $x = 2$ to get a vertex. Then one vertex is $(2, 0)$. What are the other three vertices?

   Solution: The other three vertices are $(-2, 0), (-2, 12)$ and $(2, 12)$.

   (b) What is the area of the rectangle determined by this choice $x = 2$?

   Solution: The area of the rectangle is $2 \cdot 2(16 - 2^2) = 48$.

   (c) How does the area depend on x. In other words, if R is the rectangle determined by x, (and $-x, f(x), f(-x)$), what is the area $A(x)$ of R?

   Solution: The area $A(x)$ is given by $A(x) = 2x(16 - x^2) = 32x - 2x^3$.

   (d) What choices of x give rise to rectangles? In other words, what is the domain of the function in part 3.

   Solution: One reasonable domain for this problem is the interval $[0, 4]$ since there is no reason to include negative values of x

   (e) What are the dimensions of such a rectangle with the greatest possible area?

   Solution: Compute $A'(x) = 32 - 6x^2$ and determine the critical points. Compare the values of A at the endpoints with those at the critical points to determine the maximum value of A. This turns out to be $x = \sqrt{16/3}$ and the maximum value is $A(\sqrt{16/3}) = (2\sqrt{16/3})(32/3) = 256\sqrt{3}/9 \approx 49.26$.

8. (10 points) Consider the line $y = 4$ and the point $P = (3, 2)$. For each real number x, let $D(x)$ denote the distance from the point P to the point $(x, 4)$ on the line. Find $D(x)$. State in words what it means when $D'(x) = 0$. In other words, what is the geometric meaning, not simply that x is a critical point of D. Find a critical point of D. Why does this value make sense? Write a complete sentence about your reasoning.

   Solution: $D(x) = \sqrt{(4-2)^2 + (x-3)^2} = \sqrt{4 + (x-3)^2} = \sqrt{x^2 - 6x + 13}$. Then $d'(x) = \frac{1}{2}(x^2 - 6x + 13)^{-\frac{1}{2}} \cdot [2x - 6] = \frac{x-3}{\sqrt{x^2-6x+13}}$, which has a zero at $x = 3$. $D'(x) = 0$ means that the tangent line to D is horizontal and it means here that the function D changes direction from going down to going up.

9. **Compound Interest.** (15 points)

   (a) Find the time required for a 12% investment compounded quarterly to quadruple in value. Is there a way to apply the 'rule of 72' to estimate this answer.

   Solution: Solve $4 = (1.03)^{4t}$ for t by taking logs to get $t = \frac{\log 4}{4 \log 1.03} \approx 11.72$. The rule of 72 would predict that a 12% investment takes 6 years to double, hence 12 years to quadruple.

   (b) Given that a certain investment compounded continuously has taken exactly 14 years to triple its value, what was the rate of interest?

   Solution: Solve the equation $3 = 1e^{14r}$ by taking logs of both sides to get $\ln 3 = 14r$ from which we get $r = 0.0785$.

   (c) Now change (b) so the compounding is annual and work the same problem again. Explain why this rate is higher than in part (b).

   Solution: Solve the equation $3 = (1+r)^{14}$ to get $\log(r+1) = \frac{\log 3}{14}$, so $1 + r = 10^{0.0341} \approx 1.0816$ which results in $r = 0.0816$, about 8%. The rate is higher because, it takes longer for an account compounded annually to double than one at the same rate compounded continuously.