April 12, 2010 **Name** ______________________________

The total number of points available is 145. Throughout this test, **show your work.**

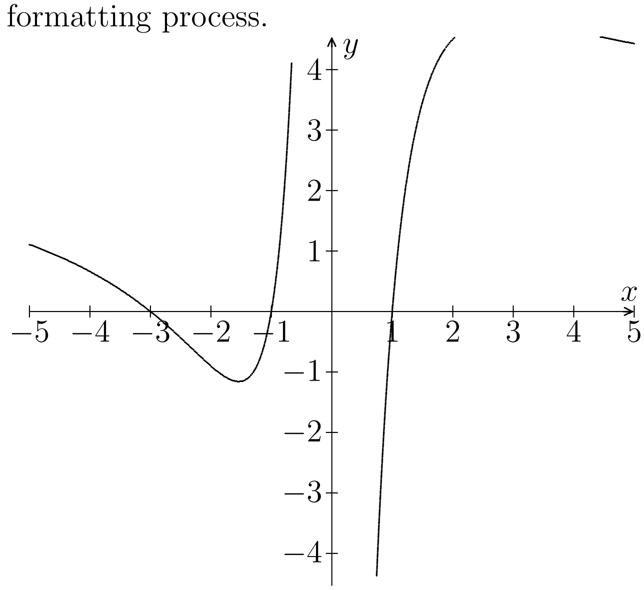
1. (15 points) Consider the function $f(x) = x^3 - 8x^2 + 3, \quad -2 \le x \le 10$. Find the locations of the absolute maximum of $f(x)$ and the absolute minimum of $f(x)$ and the value of f at these points.

   Solution: Since $f'(x) = 3x^2 - 16x$ we have one critical point at $x = 0$ and the other at $x = \frac{16}{3}$. The other two candidates for extrema are the endpoints, -2 and 10. Checking functional values, we have $f(0) = 3, f(16/3) \approx -11.22, f(-2) = -37$ and $f(10) = 203$. So f has an absolute maximum of 203 at $x = 10$ and an absolute minimum of -37 at $x = -2$.

2. (20 points) Find a rational function $r(x)$ that has exactly three zeros, $x = -3$ $x = -1$ and $x = 1$, exactly one vertical asymptote at a number between -4 and 4, and has a horizontal asymptote $y = 3$.

   (a) Sketch the graph of your $r(x)$.

   Solution:
   Notice that the graph of the function over the interval from 2 to 4 go obliterated in the formatting process.



   (b) Find a symbolic representation of r.

Solution: There are a few ways to do this. The easiest is to make the numerator $(x+3)(x^2-1)$ and the denominator either x^3 or $x(x^2+1)$. To make the function have $y = 3$ as an asymptote, we can simply multiply by 3. Thus $r(x) = \frac{3(x+3)(x^2-1)}{x^3+x}$.

3. (30 points) Consider the function $f(x) = (3x-1)^2(x-4)^2$.

   (a) Find $f'(x)$ and $f''(x)$.

   Solution: By the product rule, $f'(x) = 2(3x-1)\cdot 3(x-4)^2 + 2(x-4)(3x-1)^2 = 2(3x-1)(x-4)[3(x-4)+(3x-1)] = 2(3x-1)(x-4)(6x-13)$, and $f''(x) = 2\cdot 3(x-4)(6x-13) + \cdot 6(3x-1)(x-4) + (3x-1)(6x-13) = 2(54x^2 - 234x + 139)$.

   (b) Find all the critical points of f.

   Solution: The zeros of f' are $1/3, 13/6$, and 4. There are no singular points.

   (c) Apply the Test Interval Technique to find the sign chart for f' and use the information in the sign chart to classify the critical points of f. In other words, tell whether each one is the location of (a) a relative maximum, (b) a relative minimum, or (c) neither a relative max or min.

   Solution: From the work above, we see that the intervals are $(-\infty, 1/3), (1/3, 13/6), (13/6, 4), (4, \infty)$. The sign of f' is negative over the first and third of these and positive over the other two.

   (d) List the intervals over which f is increasing.

   Solution: Reading from the sign chart for f', we see that f is increasing over $(1/3, 13/6)$ and $(4, \infty)$.

   (e) Discuss the concavity to f and find all the inflection points on the graph of f.

   Solution: Use the quadratic formula to solve $54x^2 - 234x + 193 = 0$ to get the two roots $x \approx 1.18$ and $x \approx 3.23$. Since f'' is a parabola that opens upward, we can see that $f''(x) < 0$ between the two roots and positive outside the two. So f is concave upwards on $(\infty, 1.18)$ and on $(3.23, \infty)$ and concave downward on the interval between them. There are points of inflection at roughly $(1.18, f(1.18))$ and $(3.23, f(3.23))$.

4. (15 points) Find an equation for the line tangent to (the graph of) $f(x) = xe^{2x}$ at the point $(2, 2e^4)$. You can leave your answer in terms of e. In other words, you need not find decimal approximations.

   Solution: Since $f' = e^{2x} + 2xe^{2x}$, by the product rule, it follows that $f'(2) = e^4 + 4e^4 = 5e^4$, and that the tangent line is given by $y - 2e^4 = 5e^4(x - 2)$, which in slope-intercept form is $y = 5e^4x - 8e^4$.

5. (20 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the difference between the object's temperature and that of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

   where T is the air temperature, $72°F$, A and k are positive constants, and t is expressed in minutes.

   (a) What is the value of A?

   Solution: Since $F(t) = 72 + Ae^{-kt}$, it follows that $F(0) = 72 + A \cdot 1 = 212$ so $A = 140$.

   (b) Suppose that after exactly 11 minutes, the temperature of the coffee is $132°F$. What is the value of k?

   Solution: Solve $F(t) = 132 = 72 + 140e^{-k(11)}$ for k to get $k = 0.077$.

   (c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

   Solution: Solve the equation $80 = 72 + 140e^{-0.077t}$ for t to get first $e^{-0.077t} = 8/140 = 0.05714$, and taking logs of both sides yields $t = 37.2$ minutes.

   (d) After an hour, the coffee reaches a temperature of about $73°$. It has lost temperature at a rate of $(212 - 73)/60 \approx 7/3$ degrees per minute. Find a time t when the coffee is losing exactly $7/3$ degrees per minute.

   Solution: Since $F'(t) = -kAe^{-kt} \approx -0.077 \cdot 140 \cdot e^{-}0.077t$, we need to find the t for which $F'(t) = -7/3$ The value of t that works here is $-7/3 \cdot 1/10.78 \cdot 1/(-0.077) \approx 19.8$ minutes.

6. (10 points) The population of the world in 1990 was 5 billion and the relative growth rate was estimated at 1.5 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2020.

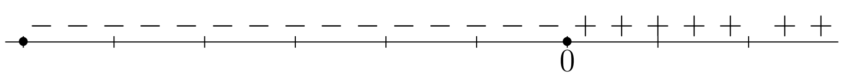
   Solution: The function is $P(t) = p_0 e^{rt}$, and we know $p_0 = P(0) = 5$ billion, so $P(30) = 5 \cdot e^{0.015(30)} = 5e^{0.45} \approx 7.84$ billion.

7. (20 points) Consider the function $f(x) = x \ln(x^2 + 2)$.

   (a) Find $f'(x)$.
   Solution: $f'(x) = \ln(x^2 + 2) + \frac{2x^2}{x^2+2}$.

   (b) Find $f''(x)$.
   Solution: $f''(x) = \frac{2x}{x^2+2} + \frac{4x(x^2+2)-2x(2x^2)}{(x^2+2)^2} = \frac{2x(x^2+6)}{(x^2+2)^2}$ after a lot of arithmetic.

   (c) Find the sign chart for $f''(x)$.
   Solution: Since $x^2 + 6$ and $x^2 + 2$ have no zeros, $f''(x) < 0$ on $(-\infty, 0)$ and positive on $(0, \infty)$, as shown on the sign chart for f'':



   (d) Find the intervals over which f is concave upwards.
   Solution: From (c) it follows that f is concave upwards on $(0, \infty)$.

8. (15 points) A rancher wants to fence in a rectangular grazing area along a river and then divide it in half with a fence down the middle perpendicular to the river to separate the cows from the sheep. What is the largest area that the rancher fence in if he has 12 miles of fencing material?

   Solution: Let x denote the lengths of the three parts that are perpendicular to the river and let y denote the length of the segment of fence parallel to the river. See the figure. The farmer wants to use all the fencing so $3x + y = 12$. Thus $y = 12 - 3x$. The farmer wants the area $A(x) = xy = x(12 - 3x) = 12x - 3x^2$ to be as large as possible. Since $A'(x) = 12 - 6x$, A has only one critical point, $x = 2$. Build the sign chart to see that $x = 2$ is the location of a maximum, and for this value, $y = 6$ and $A(2) = 2 \cdot 6 = 12$ square miles.