November 24, 2009 **Name** ______________________

The total number of points available is 145. Throughout this test, **show your work.**

1. (10 points) Find an equation for the line tangent to the graph of $y = \ln(x^2+1)$ at the point $(1, \ln(2))$.

   Solution: The derivative of the function is $y' = 2x/(x^2+1)$ so the slope of the line at $x = 1$ is $\frac{2\cdot 1}{1^2+1} = 1$ and the line is $y - \ln(2) = 1(x-1)$. That is $y = x - 1 + \ln(2)$.

2. (10 points) Find an equation for the line tangent to the graph of $y = e^{2x-3}$ at the point $(3/2, 1)$.

   Solution: $y' = 2(e^{2x-3}$ so $m = 2e^{2\cdot\frac{3}{2}-2} = 2$. Thus the line is $y-1 = 2(x-3/2)$. and in slope-intercept form $y = 2x - 2$.

3. (15 points) Let $H(x) = \ln(4x^2+12x+10) - 2x$. Find all the critical points.

   Solution: We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2 + 16x + 8 = 0$ which has repeated roots, $x = -1$.

4. (20 points) Consider the function $f(x) = (2x-4)e^{x^2}$.

   (a) Use the product rule to find $f'(x)$.
   Solution: $f'(x) = 2e^{x^2} + 2x(2x-4)e^{x^2}$

   (b) List the critical points of f.
   Solution: Factor the expression above to get $(4x^2-8x+2)e^{x^2} = 2e^{x^2}(2x^2-4x+1)$, which has value 0 when $x = 1 - \sqrt{2}/2 \approx 0.293, x = 1 + \sqrt{2}/2 \approx 1.707$. Call the first value α and the second β.

   (c) Construct the sign chart for $f'(x)$.
   Solution: f' is positive on $(-\infty, \alpha)$ and on (β, ∞).

   (d) Write in interval notation the interval(s) over which f is increasing.
   Solution: f is increasing on $(-\infty, \alpha)$ and on (β, ∞).

5. (15 points) Consider the function $f(x) = \ln[(2x^2+3)(7x-2)(x^2-4)]$.

   (a) Recalling that $\ln(x)$ is defined precisely when $x > 0$, find the domain of f.

   Solution: Build the sign chart for the function $g(x) = (2x^2+3)(7x-2)(x^2-4) = (2x^2+3)(7x-2)(x-2)(x+2)$ to see that g is positive on $(-2, 2/7)$ and $(2, \infty)$. So the domain of the function f is the union of these two sets.

   (b) Let $g(x) = (2x^2+3)(7x-2)^2(x^2-4)^3$. Use logarithmic differentiation to find g'. Find a decimal representation of $g'(1)$.

   Solution: Take logs of both sides to get $\ln g(x) = \ln(2x^2+3)(7x-2)^2(x^2-4)^3$. This simplifies to $\ln g(x) = \ln(2x^2+3) + 2\ln(7x-2) + 3\ln(x^2-4)$. Now taking the derivative of both sides yields

$$\frac{g'(x)}{g(x)} = \frac{4x}{2x^2+3} + \frac{2\cdot 7}{7x-2} + \frac{3\cdot 2x}{x^2-4}$$

   Finally, we can write

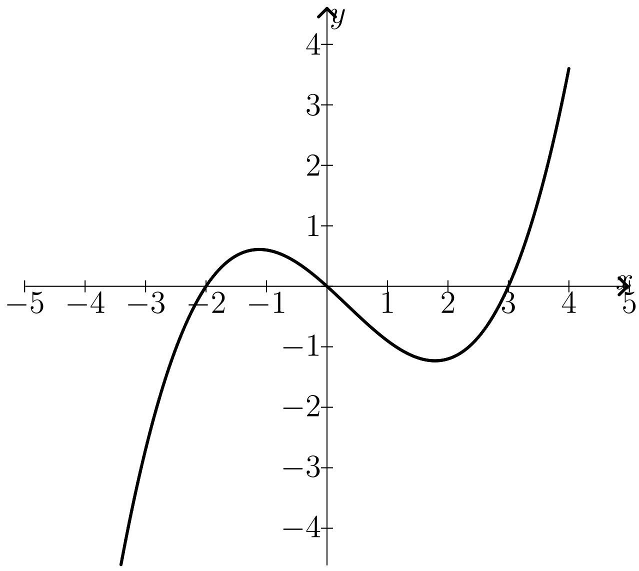
$$g'(x) = (2x^2+3)(7x-2)^2(x^2-4)^3\left(\frac{4x}{2x^2+3} + \frac{14}{7x-2} + \frac{6x}{x^2-4}\right).$$

   Thus, $g'(1) = g(1)\left(\frac{4}{2+3} + \frac{14}{7-2} + \frac{6}{1-4}\right) = 5\cdot 5^2\cdot(-3)^3(0.8+2.8-2) = -3375\cdot 1.6 = -5400$.

6. (15 points) Sketch the graph of a function defined on the interval $[-4, 4]$ with the following properties.

   - It has zeros at $x = -2, x = 0$ and $x = 3$,
   - it has horizontal tangent lines at $x = -1$ and $x = 2$, and
   - it has a relative maximum at $x = -1$, a relative minimum at $x = 2$ and an absolute maximum value of 3 at the point $x = 4$.

Solution:



7. (20 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where T is the air temperature, $68°F$, A and k are positive constants, and t is expressed in minutes.

   (a) What is the value of A?
   Solution: Since $F(t) = 68 + Ae^{-kt}$, it follows that $F(0) = 68 + A \cdot 1 = 212$ so $A = 144$.

   (b) Suppose that after exactly 8 minutes, the temperature of the coffee is $136.6°F$. What is the value of k?
   Solution: Solve $F(t) = 136.6 = 68 + 144e^{-k(8)}$ for k to get $k = 0.09269$.

   (c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.
   Solution: Solve the equation $80 = 68 + 144e^{-0.09269t}$ for t to get first $e^{-0.09269t} = 12/144 \approx 0.83333$, and taking logs of both sides yields $t =$ 26.8 minutes.

8. (30 points) Exactly 50 rabbits were introduced to a tiny South Pacific island. After one year, there were 400 rabbits and after 20 years, there were 500 rabbits. Assuming logistic growth, find the three unknown parameters and answer the questions below. You may assume $\lim_{t\to\infty} Q(t) = Q(20)$.

   (a) What is the number of rabbits after 6 months?
   Solution: The function has the form $Q(t) = \frac{A}{1+Be^{-kt}}$. The three important pieces of information are (a) $Q(0) = 50$, (b) $Q(1) = 400$, and (c) $\lim_{t\to\infty} Q(t) = Q(20) = 500$. Use (c) first to get $\frac{A}{1+0} = A = 500$ and then use (a) to get $\frac{500}{1+B} = 50$, which implies $B = 9$. Then use (b) to find that $k = \ln(36) \approx 3.5835$. Next, compute $Q(0.5) \approx 200.0$.

   (b) What is the number of rabbits after 2 years?
   Solution: Use the parameters found in the first part to find that $Q(2) = 500(1 + 9e^{-2\ln(36)})^{-1} \approx 496.5$

   (c) At what time is the rate of growth of the rabbit population the greatest?
   Solution: This hard problem requires taking the second derivative of $Q(t) = \frac{A}{1+Be^{-kt}}$. The first is $Q'(t) = -A(1 + Be^{-kt})^{-2} \cdot 9 - kBe^{-kt}) =$

$\frac{ABke^{-kt}}{(1+Be^{-kt})^2}$. Then $Q''(t) = \frac{-(1+Be^{-kt})(ABk^2e^{-kt})-2AB^2k^2e^{-kt}(1+Be^{-kt})Be^{-kt}}{(1+Be^{-kt})^4}$. After factoring the numerator N, we have $N = -ABk^2(1+Be^{-kt})(e^{-kt})[1+Be^{-kt} - 2Be^{-kt}]$. Setting $[1 + Be^{-kt} - 2Be^{-kt}]$ equal to zero yields $Be^{-kt} = 1$. This means that the value of t that maximizes $Q'(t)$ is the one for which $Q(t) = \frac{A}{1+Be^{-kt}} = \frac{A}{2}$. Solve $9e^{-\ln(36)t} = 1$ for t to get $t = \frac{\ln(9)}{\ln(36)} \approx 0.61$ years.

9. (10 points) How long does it take an investment of \$$P$ at an annual rate of 8% to triple in value if compounding

   (a) takes place quarterly? Round your answer to the nearest tenth of a year.

   Solution: We need to solve the equation $3P = P\left(1 + \frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$ for t. Using logs, we get $t = \frac{\ln 3}{4 \ln 1.02} \approx 13.87 \approx 13.9$ years.

   (b) takes place continuously? Round your answer to the nearest tenth of a year. As usual, no work shown, no credit!

   Solution: We need to solve the equation $3000 = 1000e^{0.08t}$ for t. Using logs, we get $t = \frac{\ln 3}{0.08} \approx 13.73 \approx 13.7$ years.