November 24, 2008 **Name** ______________________________

The total number of points available is 137. Throughout this test, **show your work.**

1. (15 points) Consider the function $f(x) = (x^2 - 4x + 4)e^{2x}$.
   (a) Use the product rule to find $f'(x)$.
   Solution: $f'(x) = (2x-4)e^{2x} + 2(x^2 - 4x + 4)e^{2x}$
   (b) List the critical points of f.
   Solution: Factor the expression above to get $(2x^2 - 8x + 8 + 2x - 4)e^{2x} = 2e^{2x}(x-2)(x-1)$, which has value 0 when $x = 2, x = 1$.
   (c) Construct the sign chart for $f'(x)$.
   Solution: f' is positive on $(-\infty, 1)$ and on $(2, \infty)$.
   (d) Write in interval notation the interval(s) over which f is increasing.
   Solution: f is increasing on $(-\infty, 1)$ and on $(2, \infty)$.

2. (15 points) Consider the function $f(x) = \ln[(2x^2 + 3)(7x - 2)(x^2 - 4)]$.
   (a) Recalling that $\ln(x)$ is defined precisely when $x > 0$, find the domain of f.
   Solution: Build the sign chart for the function $g(x) = (2x^2+3)(7x-2)(x^2-4) = (2x^2+3)(7x-2)(x-2)(x+2)$ to see that g is positive on $(-2, 2/7)$ and $(2, \infty)$. So the domain of the function f is the union of these two sets.
   (b) Let $g(x) = (2x^2+3)(7x-2)^2(x^2-4)^3$. Use logarithmic differentiation to find g'. You need not simplify your answer.
   Solution: Take logs of both sides to get $\ln g(x) = \ln(2x^2+3)(7x-2)^2(x^2-4)^3$. This simplifies to $\ln g(x) = \ln(2x^2+3) + 2\ln(7x-2) + 3\ln(x^2-4)$. Now taking the derivative of both sides yields

$$\frac{g'(x)}{g(x)} = \frac{4x}{2x^2+3} + \frac{2 \cdot 7}{7x-2} + \frac{3 \cdot 2x}{x^2-4}$$

   Finally, we can write

$$g'(x) = (2x^2+3)(7x-2)^2(x^2-4)^3 \left(\frac{4x}{2x^2+3} + \frac{14}{7x-2} + \frac{6x}{x^2-4} \right).$$

3. (15 points) Consider the function $f(x) = \ln(x^2 + 4)$.

(a) Find the domain of f.

Solution: f is defined for all x because $x^2 + 4$ is positive for all x.

(b) Find both $f'(x)$ and $f''(x)$.

Solution: $f'(x) = \frac{2x}{x^2+4}$ and, by the quotient rule, $f''(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2}$, which simplifies to

$$f''(x) = \frac{-2x^2 + 8}{(x^2 + 4)^2}.$$

So $f''(x)$ has two zeros, $x = 2$ and $x = -2$.

(c) Build the sign chart for $f''(x)$.

Solution: The sign chart shows that $f''(x)$ is positive only between -2 and 2.

(d) Does f have an absolute maximum or and absolute minimum? Is so, which? ... and where?

Solution: f has an absolute minimum at $x = 0$. $f(0) = \ln 4$.

(e) Find an interval over which f is increasing. As usual, to get credit for this, you must show your work.

Solution: Since f' is positive on $(0, \infty)$, it follows that f is increasing on that interval.

(f) Does f have any inflection points?

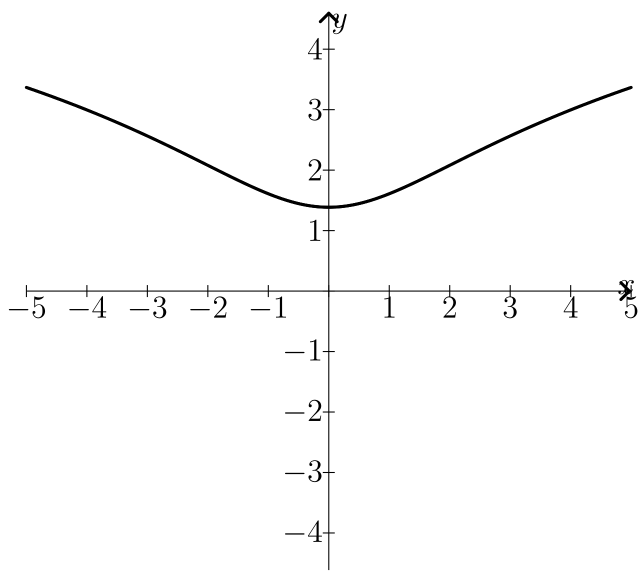
Solution: Yes, at both $(-2, \ln 8)$ and $(2, \ln 8)$.

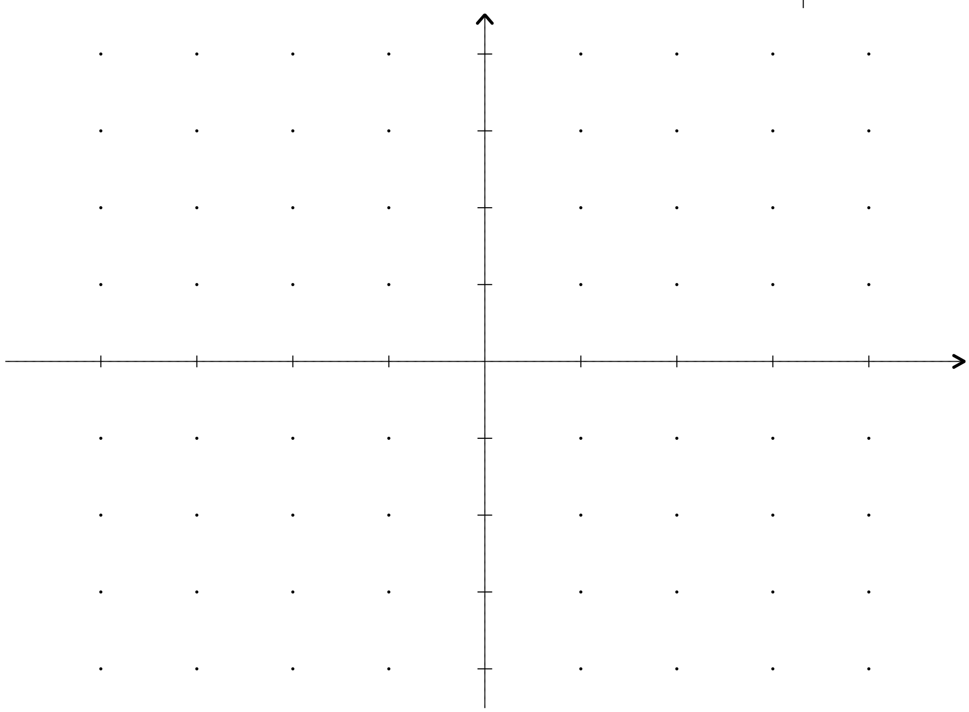
(g) Use the information from part b. to find the intervals where f is concave upwards.

Solution: Since f'' is positive on $(-2, 2)$, f is concave upwards on that interval.

(h) Sketch the graph of f.

Solution:





4. (20 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where T is the air temperature, $68°F$, A and k are positive constants, and t

is expressed in minutes.

(a) What is the value of A?

Solution: Since $F(t) = 68+Ae^{-kt}$, it follows that $F(0) = 68+A\cdot 1 = 212$ so $A = 144$.

(b) Suppose that after exactly 8 minutes, the temperature of the coffee is $136.6°F$. What is the value of k?

Solution: Solve $F(t) = 136.6 = 68 + 144e^{-k(8)}$ for k to get $k = 0.09269$.

(c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

Solution: Solve the equation $80 = 68 + 144e^{-0.09269t}$ for t to get first $e^{-0.09269t} = 12/144 = 0.83333$, and taking logs of both sides yields $t =$ 26.8 minutes.

5. (10 points) Find an equation for the line tangent to the graph of $y = \ln x^2$ at the point $(1, 0)$.

Solution: The derivative of the function is $y' = 2/x$ so the slope of the line at $x = 1$ is $2/1 = 2$ and the line is $y - 0 = 2(x - 1)$. That is $y = 2x - 2$.

6. (10 points) Find an equation for the line tangent to the graph of $y = e^{5x-2}$ at the point $(2/5, 1)$.

Solution: $y' = 5(e^{5x-2}$ so $m = 5e^{5\cdot\frac{2}{5}-2} = 5$. Thus the line is $y-1 = 5(x-2/5)$. and in slope-intercept form $y = 5x - 1$.

7. (12 points) A skull from an archeological dig has one-twelfth the amount of Carbon-14 it had when the specimen was alive.

(a) Recall that the half-life of Carbon-14 is 5770 years. Find the decay constant k.

Solution: We must solve the equation $Q(5770) = Q_0/2$ for k, where $Q(t) = Q_0e^{-kt}$. The equation leads to $e^{-5770k} = 1/2$ which means that $k \approx 1.201 \times 10^{-4} = 0.00012$.

(b) What is the age of the specimen? Round off your answer to the nearest multiple of one hundred years.

Solution: To solve the equation $Q_0e^{-kt} = Q_0/12$, divide both sides by Q_0 to get $e^{-kt} = 1/12$. This happens for $t = 20685$ years, which rounds to 20700.

8. (10 points) How long does it take an investment of \$$P$ at an annual rate of 8% to triple in value if compounding

(a) takes place quarterly? Round your answer to the nearest tenth of a year.

Solution: We need to solve the equation $3P = P\left(1 + \frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$ for t. Using logs, we get $t = \frac{\ln 3}{4 \ln 1.02} \approx 13.87 \approx 13.9$ years.

(b) takes place continuously? Round your answer to the nearest tenth of a year. As usual, now work shown, no credit!

Solution: We need to solve the equation $3000 = 1000e^{0.08t}$ for t. Using logs, we get $t = \frac{\ln 3}{0.08} \approx 13.73 \approx 13.7$ years.

9. (30 points) Spreading of a rumor. Three hundred college students attend a lecture of the Dean at which she hints that the college will become coed. The rumor spreads according to the logistic curve

$$Q(t) = \frac{3000}{1 + Be^{-kt}},$$

where t is measured in hours.

(a) Compute the parameter B.

Solution: Note that $Q(0) = 300 = \frac{3000}{1+Be^{-k0}}$, so $1 + B = 10$, and $B = 9$.

(b) How many students attend the college? Hint: the question is not 'how many attended the lecture?'

Solution: Take the limit as $t \to \infty$ since eventually everyone knows the rumor. $\lim_{t\to\infty} \frac{3000}{1+9e^{-kt}} = 3000$.

(c) Two hours after the speech, 600 students had heard the rumor. How many students had heard the rumor after four hours?

Solution: Solve $Q(2) = 600 = \frac{3000}{1+9e^{-2k}}$ for k. We get $9e^{-2k} = 4$ and $k = \frac{\ln 4 - \ln 9}{-2} \approx 0.40546$. Therefore $Q(4) = \frac{3000}{1+9e^{-4k}} = 1080$.

(d) How fast is the rumor spreading after four hours?

Solution: We need $Q'(t)$. So write $Q(t) = 3000(1 + 9e^{-kt})^{-1}$ and use the chain rule to get $Q'(t) = \frac{3000 \cdot 9ke^{-kt}}{(1+9e^{-kt})^2}$. Using a calculator, we find that $Q'(4) = \frac{27000ke^{-4k}}{(1+9e^{-4k})^2} \approx \frac{2162.48}{7.716} \approx 280$ students per hour.

(e) After how many hours will 2000 students have heard the rumor?

Solution: We need to solve the equation $Q(t) = 2000 = \frac{3000}{1+9e^{-kt}}$ for t. We have $1 + 9e^{-kt} = 3/2$, and it follows that $e^{-kt} = 1/18$. We can solve this for t to get $t = \frac{\ln 1 - \ln 18}{-k} = \frac{\ln 18}{k} \approx 7.128$ hours.