April 23, 2004 **Name** ______________________________

The total number of points available is 120. Throughout this test, **show your work.**

1. (15 points) How long does it take an 8% investment to triple if

   (a) Compounding takes place monthly?

   Solution: To find a value of t for which $3 = 1(1 + \frac{0.08}{12})^{12t}$, take ln of both sides to get $12t \ln(1 + 2/300) = \ln 3$. The value of t satisfying this equation is $t \approx 13.778$ years.

   (b) Compounding takes place continuously?

   Solution: We need to solve $3 = 1e^{0.08t}$. This is easy if we take ln of both sides. We get $0.08t = \ln 3$ which yields $t \approx 13.732$ years, just a little less than the answer to (a), as expected.

2. (15 points) Let $f(x) = x^4 + 2x^3 - 12x^2 + x - 5$.

   (a) Find the interval(s) where f is concave upward.

   Solution: $f'(x) = 4x^3 + 6x^2 - 24x + 1$ and $f''(x) = 12x^2 + 12x - 24$, which has two zeros, $x = -2$ and $x = 1$. So f'' is positive over the intervals $(-\infty, -2$ and $(1, \infty)$.

   (b) Find the inflection points of f, if there are any.

   Solution: There are two inflection points, $(-2, f(-2)) = (-2, -55)$ and $(1, f(1)) = (1, -13)$

3. (15 points) Find the absolute maximum value and the absolute minimum value of the function $f(x) = x^3 - 6x^2 + 8x + 7$ on the interval $0 \leq x \leq 6$.

Solution: Note that $f'(x) = 3x^2 - 12x + 8$, so we get the critical points using the quadratic formula:

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4 \cdot 3 \cdot 8}}{2 \cdot 3} = 2 \pm 2\sqrt{3}/3.$$

Thus we need to compare the four numbers $f(0)$, $f(2+2\sqrt{3}/3)$, $f(2-2\sqrt{3}/2)$, and $f(6)$. These values are $f(0) = 7$, $f(2+2\sqrt{3}/3) \approx 3.9207$, $f(2-2\sqrt{3}/2) \approx 9.2945$, and $f(6) = 55$. So the maximum of 55 occurs at $x = 6$ and the minimum of about 3.9207 at the point $x = 2 + 2\sqrt{3}/3$.

4. (15 points) Suppose the function $Q(t) = Q_0 e^{-kt}$ satisfies $Q(5770) = Q_0/2$.

   (a) What is the value of k?
   Solution: The number k satisfies $Q_0/2 = Q_0 e^{-k(5770)}$. That is $e^{-5770k} = 0.5$. Take ln of both sides to get $-5770k = \ln 0.5 \approx -0.693$ so $k \approx 1.2019 \times 10^{-4} = 0.00012019$.

   (b) For what value of t is it true that $Q(t) = Q_0/4$?
   Solution: Since the half-life is 5770, the quarter-life is 11540.

   (c) Find $Q'(t)$.
   Solution: $Q'(t) = -kQ_0 e^{-kt}$.

   (d) What is the rate of growth of $Q(t)$ at $t = 11540$?
   Solution: $Q'(11540) = -kQ(11540) = -kQ_0/4 \approx -3.0032 \times 10^{-5} Q_0 = -0.000030032 Q_0$.

5. (15 points) Find the interval(s) where $f(x) = (x-4)(x^2-1)(x+3)$ is positive.

Solution: The branch points are $x = 4, 1, -1$, and -3. I picked test points $-4, -2, 0$, and 5, and found that $f(-4) > 0$, $f(-2) < 0$, $f(0) > 0$, $f(2) < 0$, and $f(5) > 0$. Therefore, the function f is positive on $(-\infty, -3), (-1, 1)$, and $(4, \infty)$.

6. (15 points) Compute each of the following derivatives.

(a) $\frac{d}{dx}\sqrt{x^3+1}$

Solution: $\frac{d}{dx}(x^3+1)^{1/2} = \frac{1}{2}\left(x^3+1\right)^{-1/2} \cdot 3x^2 = \frac{3}{2}x^2(x^3+1)^{-1/2}$.

(b) $\frac{d}{dx}\ln(x^3+1)$

Solution: $\frac{d}{dx}\ln(x^3+1) = \frac{3x^2}{x^3+1}$.

(c) $\dfrac{d}{dx}\dfrac{e^x}{x}$

Solution: $\dfrac{d}{dx}\dfrac{e^x}{x} = \dfrac{xe^x - e^x}{x^2}$.

(d) $\dfrac{d}{dx}(x-3)^3(x^2+1)^4(x-8)^{12}$

Solution: Use logarithmic differentiation. First, take ln of both sides to get $\ln y = \ln(x-3)^3(x^2+1)^4(x-8)^{12} = 3\ln(x-3) + 4\ln(x^2+1) + 12\ln(x-8)$. Then differentiate both sides to get

$$\frac{y'}{y} = 3\frac{1}{x-3} + 4\frac{2x}{x^2+1} + 12\frac{1}{x-8},$$

so $y' = y\left(\frac{3}{x-3} + \frac{8x}{x^2+1} + \frac{12}{x-8}\right)$.

7. (15 points) The quantity demanded each month of the Sicard wristwatch is related to the price by the equation

$$p = \frac{50}{0.01x^2 + 1}$$

for $0 \leq x \leq 20$ where p is measured in dollars and x is measured in units of a thousand.

(a) Find the demand when the price is set at \$25 per watch.

Solution: Solve the equation $25 = \frac{50}{0.01x^2+1}$ to get $x = 10$.

(b) Recall the revenue function is the product of the price and the number of units sold. Find the revenue function $R(x)$.

Solution: $R(x) = xp = \frac{50x}{0.01x^2+1}$

(c) Use the results of part b. to find the number of (thousands of) units needed to maximize the revenue.

Solution: Note that $R'(x) = xp = \frac{50(0.01x^2+1)-0.02x(50x)}{(0.01x^2+1)^2}$, which has only one zero in the interval $[0, 20]$, and that critical point is $x = 10$. Note that $R'(10+) < 0$ and $R'(10-) > 0$ so the first derivative test tells us that R has a maximum at $x = 10$. Note that $R(10) = 10 \cdot 25 = 250$, in thousands of dollars.

8. (15 points) Four identical $x \times x$ square corners are cut from a 14×18 inch rectangular piece of metal, and the sides are folded upward to build a box.

   (a) What is the volume of the box that results when the corners cut are 1×1.

   Solution: The volume is the product length $\times$ width $\times$ height $= 12 \times 16 \times 1 = 192$.

   (b) Let $V(x)$ denote the volume of the box when the $x \times x$ corners are removed. Find $V(2)$ and $V(3)$.

   Solution: Note that $V(x) = (14-2x)(18-2x)(x)$ and $V(2) = 10 \cdot 14 \cdot 2 = 280$ and $V(3) = 8 \cdot 12 \cdot 3 = 288$.

   (c) What is the implied domain of V?

   Solution: The domain of V is $[0, 7]$.

   (d) Find $V'(x)$.

   Solution: $V'(x) = \frac{d}{dx}(14-2x)(18x-2x^2) = -2(18x-2x^2) + (18-4x)(14-2x) = 4(3x^2 - 32x + 63)$.

   (e) Find the critical points of $V(x)$.

   Solution: Use the quadratic formula to get $x = \frac{32 \pm \sqrt{32^2 - 4 \cdot 3 \cdot 63}}{6}$, one of which belongs to the domain. This number, $x = \frac{32 - \sqrt{268}}{6} \approx 2.604$.

   (f) What value of x makes the value of V maximum? Estimate within .01 the maximum value of V.

   Solution: $V(2.604) \approx 292.864 \approx 292.86$.