April 11, 2003 **Name** ____________________________

The first 6 problems count 5 points each. Problems 6 through 9 count as marked. In the multiple choice section, circle the correct choice (or choices). The total number of points available is 120.

Each of the next few items are true-false. To get full credit you must give a valid reason for your answer. Circle either True or False, and give your reason in the space provided. Generally, 2 points for the right t/f value and 3 points for the right reason.

1. True or false. If $f''(x) < 0$ on the interval (a, c) and $f''(x) > 0$ on the interval (c, b), then the point $(c, f(c))$ is a point of inflection of f.

   Solution: True. This is basically the definition of inflection point.

2. True or false. If $f'(c) = 0$, then f has a relative maximum or a relative minimum at $x = c$.

   Solution: False. The function can have neither a max nor a min at a stationary point. Look at $f(x) = x^3$ and $x = 0$.

3. True or false. If f has a relative maximum at $x = c$, then $f'(c) = 0$.

   Solution: False. All we can tell is that c is a critical point. It might be a singular point.

4. True or false. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.

   Solution: True. This is just the second derivative test.

5. True or false. If $h(x) = \sqrt{x^2 - 4}$, then $h'(x) = \frac{1}{2}(x^2 - 4)^{-1/2}$.

   Solution: False. Look up the chain rule.

6. True or false. The function $g(x) = (x - 1)^{2/3}$ has a singular point at $x = 1$.

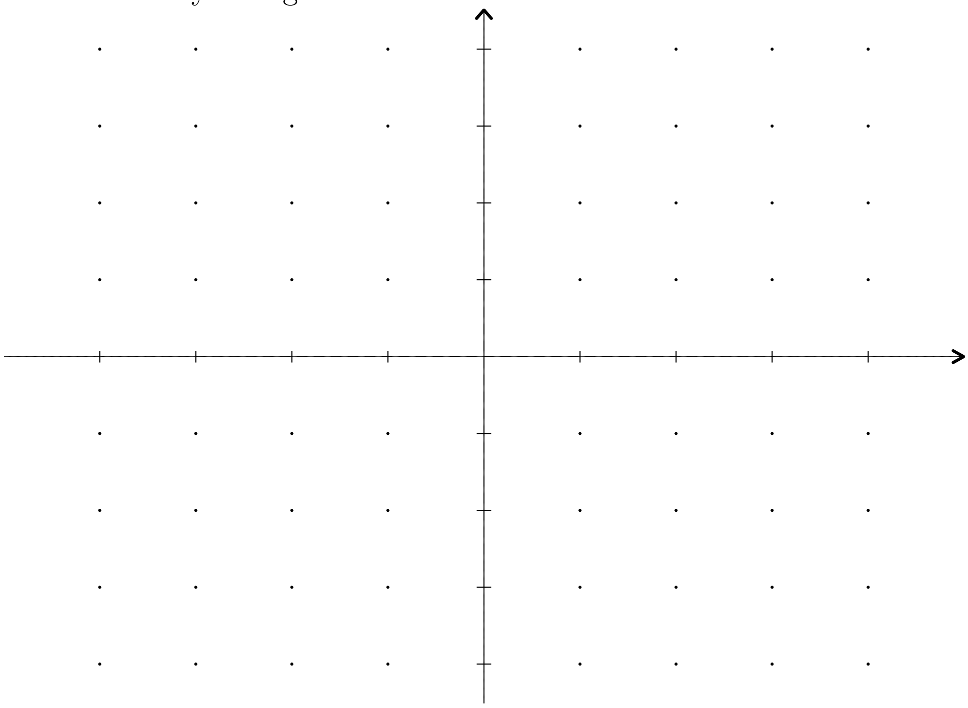
   Solution: True. Since $g'(x) = 2(x - 1)^{-1/3}/3 = 2/3(x - 1)^{-1/3}$, you can see that $g'(1)$ is not a number. Therefore $x = 1$ is a singular point.

On all the following questions, **show your work.**

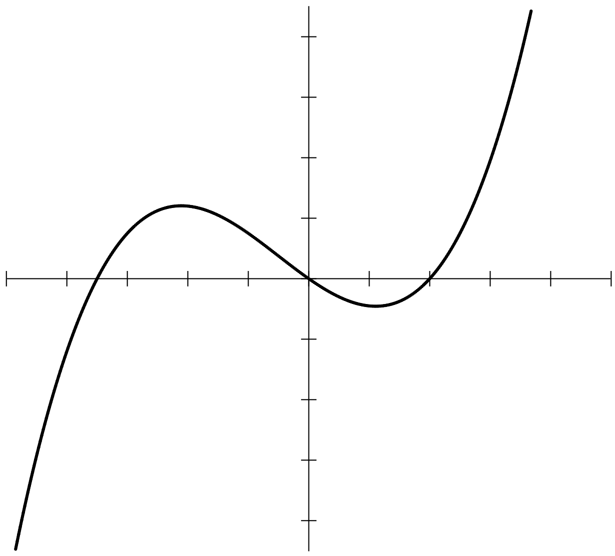
7. (20 points) Sketch the graph of a function $g(x)$ satisfying the properties shown in the table below.

x	$g(x)$	$g'(x)$
-2	1	0
0	0	-1
2	0	1

Use the coordinate system given.



Solution: One possible graph is



8. (20 points) Let $g(x) = (2x-4)^2(x+3)^2$.

   (a) Use the test interval technique (not a graphing calculator) to find the intervals over which g is increasing.

   Solution: First, note that $g'(x) = 2(2x-4)(x+3)[2x+6+2x-4] = 2(2x-4)(x+3)(4x+2)$, so the critical points are $x=-3, x=-1/2$, and $x=2$. Using the test interval technique on the intervals $(-\infty,-3), (-3,-1/2), (-1/2,2)$ and $(2,\infty)$ with test points $-4,-1,0$, and 3, we can see that

test point	2x-4	x+3	4x+2	sign of g'
-4	$-$	$-$	$-$	$-$
-1	$-$	$+$	$-$	$+$
0	$-$	$+$	$+$	$-$
3	$+$	$+$	$+$	$+$

   We conclude that $g(x)$ is increasing on the two intervals $(-3,-1/2)$ and $(2,\infty)$.

   (b) Find and classify each critical point as a location of a. a relative maximum, b. a relative minimum, or c. neither a relative max nor a relative min.

   Solution: We can use the second derivative test on the three critical points of g. Since $g''(x) = 2\cdot 2(x+3)(4x+2)+2(2x-4)\cdot\frac{d}{dx}((x+3)(4x+2))$, by the product rule, it follows that $g''(x) = 2\cdot 2(x+3)(4x+2)+2(2x-4)(4x+2+4(x+3))$. Therefore $g''(-3)=200, g''(-1/2)=-100$, and $g''(2)=200$. The second derivative test now confirms what we felt, that is, there is a relative maximum at $-1/2$ and relative minimums at both -3 and 2.

9. (15 points) Consider the rational function

$$f(x) = \frac{(x^2-4)(2x+1)}{(3x^2-3)(x-2)}.$$

   (a) Find the horizontal asymptote(s).

   Solution: The coefficient of x^3 in the numerator is 2 while that in the denominator is 3, so $y = 2/3$ is the horizontal asymptote.

   (b) Find the vertical asymptotes.

   Solution: To find the vertical asymptotes, you must first reduce the fraction to lowest terms, which mean cancelling out the common factors, in this case, just the $x-2$'s. This results in a denominator that has value 0 only at $x = \pm 1$, so these are the two vertical asymptotes.

(c) Compute $\lim_{x\to\infty} f(x)$.

Solution: The limit in question is the same as the horizontal asymptote, 2/3.

10. (15 points) Four congruent $x \times x$ squares from the corners of a cardboard rectangle that measures 16×12. The sides are then folded upward to form a topless box.

(a) Find the volume V as a function of x. What is the logical domain?

Solution: $V(x) = (16-2x)(12-2x)x$.

(b) Compute $V(0), V(1), V(2)$, and $V(3)$.

Solution: $V(0) = 0, V(1) = 14 \cdot 10 \cdot 1 = 140, V(2)12 \cdot 8 \cdot 2 = 192$, and $V(3) = 10 \cdot 6 \cdot 3 = 180$.

(c) Find $V'(x)$.

Solution:

$$\begin{aligned} V'(x) &= -2(12-2x)x + (16-2x)(\frac{d}{dx}(12-2x)x) \\ &= -2(12-2x)x + (16-2x)[(-2)x + 1(12-2x)] \\ &= -24x + 4x^2 + (16-2x)(-4x+12) \\ &= 12x^2 - 112x + 192 \end{aligned}$$

(d) Use the results from the question above to determine the critical points of V.

Solution: Use the quadratic formula to find that the critical point is $c = \frac{14-2\sqrt{13}}{3} \approx 2.263$. The other root of the quadratic is outside the domain of V.

(e) Find the absolute maximum value of V and the value of x where it occurs.

Solution: Simply evaluate V at the two endpoints and the critical point to see that the maximum value of V is $V(c) \approx 194.07$.

11. (20 points) Compute each of the following derivatives.

   (a) $\frac{d}{dx}\sqrt{x^3+1}$
   Solution: $\frac{d}{dx}\sqrt{x^3+1} = \frac{3x^2}{2\sqrt{x^3+1}}$.

   (b) $\frac{d}{dx}(2x^2+1)^{10}$
   Solution: $\frac{d}{dx}(2x^2+1)^{10} = 40x(2x^2+1)^9$.

   (c) $\dfrac{d}{dx}\left(\dfrac{2x+1}{x^2+1}\right)$
   Solution: $\dfrac{d}{dx}\left(\dfrac{2x+1}{x^2+1}\right) = -\dfrac{2(x^2+x-1)}{(x^2+1)^2}$.

   (d) $\frac{d}{dx}(2x^2+1)(3x-4)$
   Solution: $\frac{d}{dx}(2x^2+1)(3x-4) = 4x(3x-4)+3(2x^2+1) = 18x^2-16x+3$.