April 10, 2019 **Name** ______________________________

The problems count as marked. The total number of points available is xxx. Throughout this test, **SHOW YOUR WORK.**

1. (15 points) Consider the cubic curve $f(x) = 2x^3 + 3x^2 - 36x + 17$.

   (a) Build the sign chart for $f'(x)$.
   Solution: $f'(x) = 6x^2+6x-36 = 6(x^2+x-6) = 6(x+3)(x-2)$, which is negative over $(-3, 2)$ and positive elsewhere.

   (b) Using the sign chart for $f'(x)$, find the intervals over which $f(x)$ is increasing. .
   Solution: Since $f'(x)$ is positive on $(-\infty, -3)$ and on $(2, \infty)$, f is increasing over those intervals.

   (c) Find a point of inflection on the graph of f.
   Solution: $f''(x) = 12x+6 = 6(2x+1)$, which changes signs at $x = -1/2$, so there is a point of inflection at $(-1/2, f(-1/2)) = (-0.5, 35.5)$.

2. (15 points) Consider the cubic curve $g(x) = e^{x^3-12x}$.

   (a) Find $g'(x)$ and $g''(x)$.
   Solution: $g'(x) = e^{x^3-12x}(3x^2-12)$ and $g''(x) = e^{x^3-12x}[(3x^2-12)^2+6x]$.

   (b) Build the sign chart for $g''(x)$.
   Solution: To Build the sign chart for $g''(x)$, we must locate the zeros of g''. Note that $(3x^2 - 12)^2$ is zero only at $x = \pm 2$, and $6x > 0$ near $x = 2$. So we can be sure that there are just two zeros of g'' and they are both near $x = -2$. Note that $g''(-2) = -12$. Use the IVT to get zeros of $g''(x)$ near -2.3 and -1.7. Full credit awarded for any values in the range $(-2.4, -2.1)$ and $(-1.9, -1.6)$.

   (c) Use the information in (b) to discuss the concavity of g. No points for a bold answer without reference to the sign chart.
   Solution: Since $g''(x)$ is negative only when x is roughly in $(-2.28, -1.72)$, g must be concave upwards over the rest of the real numbers.

3. (30 points) Find the critical points for each of the functions given below. For credit, you must show the equation you're solving to get the critical points.

   (a) $f(x) = (x - 3)^{\frac{2}{3}}$.
   Solution: $f'(x) = \frac{2}{3}(x - 3)^{-1/3}$, so $x = 3$ is a singular point.

   (b) $g(x) = \ln(x^3 - 3x + 22)$.
   Solution: $g'(x) = \frac{3x^2-3}{x^3-3x+22}$ so $x = \pm 1$ are stationary points.

(c) $h(x) = \left(\frac{2x-1}{3x+1}\right)^4$

Solution: $h'(x) = 4\left(\frac{2x-1}{3x+1}\right)^3 \cdot \frac{2(3x+1)-3(2x-1)}{(3x+1)^2}$. This function has just one zero, at $x = -1/2$, and no singular points because h is not defined at $x = -1/3$.

(d) $f(x) = e^{2x} - 5x$

Solution: $f'(x) = 2e^{2x} - 5$ has just one zero, $x = \frac{\ln 5 - \ln 2}{2} \approx 0.458$

(e) $k(x) = \ln(6x^2 + 5x + 2) - x$.

Solution: $k'(x) = \frac{12x+5}{6x^2+5x+2} - 1$. Setting this equal to zero yields $12x + 5 = 6x^2 + 5x + 2$, which is equivalent to $6x^2 - 7x - 3 = (3x+1)(2x-3)0$, so the critical points are $x = -1/$ and $x = 3/2$.

4. (15 points) Meliha invests \$1000 at a rate of r percent compounded continuously. After 16 years her investment is worth \$4000.

   (a) How long did it take for her \$1000 investment to double?

   Solution: The doubling time must be 8 years since the investment doubles twice in 16 years.

   (b) How long did it take her investment to triple?

   Solution: Solve $2P = Pe^{rt}$ for r when $t = 8$, so get $8r = \ln 2$ or $r = \ln 2/8 \approx 8.66\%$. Then solve $3P = Pe^{rt}$ for t when $r = \ln 2/8$, so get $t = 8\ln 3 \div \ln 2 \approx 12.68$ years.

5. (15 points) Rachel learns typing in a 14 week class. The number of words per minute Rachel can type after t weeks is given by

$$F(t) = 120 - 40e^{-.4t}$$

   (a) How many weeks into the course does it take for Rachel to reach a speed of 100 words per minute.

   Solution: Solve $120 - 40e^{-.4t} = 100$ to get $t = 1.73$ weeks.

   (b) During the third week of the class, at what rate is Rachel's typing speed increasing?

   Solution: $F'(t) = 16e^{-0.4t}$, so $F'(3) = 16e^{-1.2} \approx 4.8$ words per minute. Or $F'(2.5) = 5.88$. Alternatively, calculate $F(3) - F(2) \approx 5.9$.

6. (10 points) The population of the world in 1990 was 5 billion and the relative growth rate was estimated at 1.5 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2010.

   Solution: The function is $P(t) = p_0 e^{rt}$, and we know $P(0) = 5$ billion, so $P(20) = 5 \cdot e^{0.15(20)} = 5e^{0.3} \approx 6.749$ billion.

7. (10 points) Let $g(x) = x\ln(x)$. Notice that $g(e) = e\ln(e) = e$. Find an equation for the line tangent to g at the point (e, e).

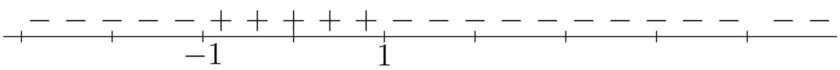
   Solution: First note that, by the product rule, $g'(x) = \ln(x) + 1$. So $g'(e) = \ln(e) + 1 = 2$. Hence the line is $y - e = 2(x - e)$ which is $y = 2x - e$.

8. (20 points) Consider the function $f(x) = \ln(x^2 + 1)$.

   (a) Find $f'(x)$.
   Solution: $f'(x) = \frac{2x}{x^2+1}$.

   (b) Find $f''(x)$.
   Solution: $f''(x) = \frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$.

   (c) Find the sign chart for $f''(x)$.
   Solution: $f''(x) < 0$ on $(-\infty, -1)$ and on $(1, \infty)$ and positive on $(-1, 1)$, as shown on the sign chart for f'':



   (d) Find the intervals over which f is concave upwards.
   Solution: From (c) it follows that f is concave upwards on $(-1, 1)$.