Math 1120 **Calculus** **Test 3.**

November 4, 1999 **Name** ______________________

In the first 10 problems, each part counts 5 points (total 50 points) and the final three problems count 20 points each.

Multiple choice section. Circle the correct choice. You do not need to show your work on these problems.

1. Which one of the following points is a singular point of

$$f(x) = (x-1)^{2/3}?$$

(A) -1 **(B)** 0 **(C)** $\boxed{1}$ **(D)** 2 **(E)** 5

2. Which of the following points is a stationary point of

$$f(x) = 3x^3 - 4x^2 - 5x + 6?$$

(A) $\dfrac{4+\sqrt{29}}{9}$ **(B)** $\boxed{\frac{4+\sqrt{61}}{9}}$ **(C)** $\dfrac{8-\sqrt{61}}{9}$ **(D)** $\dfrac{4-\sqrt{116}}{9}$ **(E)** 7

Solution. $f'(x) = 9x^2 - 8x - 5$. Use the quadratic formula to solve $f'(x) = 0$.

$$x = \frac{8 \pm \sqrt{8^2 + 4\cdot 9 \cdot 5}}{2\cdot 9} = \frac{4 \pm \sqrt{61}}{9}$$

3. Which one of the following is a point of inflection of $h(x) = 48x^2 - \frac{1}{x} - 12$?

(A) $(-1, 35)$ **(B)** $\boxed{\left(-\frac{1}{2}, -4\right)}$ **(C)** $\left(\dfrac{1}{2}, -4\right)$

(D) $(1, 35)$ **(E)** the function h has no points of inflection

3b. Which one of the following is a point of inflection of $h(x) = 8x^2 - \frac{1}{x} + 3$?

(A) $(1, 10)$ **(B)** $(-1, 12)$ **(C)** $(2, 34.5)$

(D) $\boxed{\left(\frac{1}{2}, 3\right)}$ **(E)** the function h has no points of inflection

4. Which of the following is a horizontal asymptote of

$$r(x) = \frac{(2x-3)^2(6x+1)}{(3x+2)(2x^2+1)}?$$

(A) $y = 0$ **(B)** $y = \frac{2}{3}$ **(C)** $y = 2$ **(D)** $\boxed{y = 4}$ **(E)** $y = 6$

4b. Which of the following is a horizontal asymptote of

$$r(x) = \frac{(2x-3)^2(6x+1)}{(3x+2)^2(2x^2+1)}?$$

(A) $\boxed{y = 0}$ **(B)** $y = \frac{2}{3}$ **(C)** $y = 2$ **(D)** $y = 4$ **(E)** $y = 6$

5. Let $h(x) = \dfrac{(x-1)^2(2x+2)(x+4)}{(x-2)^2(x+3)(x-5)}$. Which one of the following lines is **not** an asymptote of h?

(A) $y = 2$ **(B)** $\boxed{x = 1}$ **(C)** $x = 2$ **(D)** $x = 5$ **(E)** $x = -3$

6. Over which of the intervals is the function defined by $f(x) = x^3+3x^2-24x+18$ increasing?

(A) $[-5, -2]$ **(B)** $[-3, -1]$ **(C)** $[-1, 2]$ **(D)** $[1, 4]$ **(E)** $\boxed{[3, 7]}$

6b. Over which of the intervals is the function defined by $f(x) = x^4 - 6x^3 + 12x^2 - 24x + 18$ concave downwards?

(A) $(-5, -4)$ **(B)** $(-3, -2)$ **(C)** $(-1, 0)$ **(D)** $\boxed{(1, 2)}$ **(E)** $(3, 5)$

Solution. The second derivative is given by $f''(x) = 12(x-2)(x-1)$, so $f''(x) < 0$ precisely for $1 < x < 2$.

7b Consider the function f defined symbolically by

$$f(x) = \begin{cases} (x-1)(x-2)(x-3)+1 & \text{if } x \le 0 \\ (1/3x^{-3}) + 2/3 & \text{if } x > 0 \end{cases}$$

Notice that $f(1) = 1$. Find the y-intercept of the line tangent to the graph of f at the point $(1, 1)$.

(A) -1 **(B)** $\boxed{0}$ **(C)** 1 **(D)** 2 **(E)** None of A, B, C, and D

Solution. The function is just $f(x) = x^3/3$ and its derivative is given by $f'(x) = x^2$, so $f'(1) = 1$ and the tangent line is $y - 1 = 1(x - 1)$ which goes through the point $(0, 0)$.

7. What is $\lim_{h\to 0} \dfrac{\sqrt{7+3h} - \sqrt{7}}{h}$?

(A) $-1/\sqrt{7}$ **(B)** $-3/\sqrt{7}$ **(C)** $-3/2\sqrt{7}$ **(D)** $3/\sqrt{7}$ **(E)** $\boxed{3/2\sqrt{7}}$

8. What is $\lim_{h\to -\infty} \dfrac{x^3 - 3x^2 + 10}{3|x^3| + 2x}$?

(A) -3 **(B)** $\boxed{-1/3}$ **(C)** 0 **(D)** 1/3 **(E)** The limit does not exist

9. Suppose the functions f and g are differentiable and their values at certain points are given in the table. The next two problems refer to these functions f and g. Recall that, for example, the entry 1 in the first row and third column means that $f'(0) = 1$.

x	$f(x)$	$f'(x)$
0	2	1
1	7	3
2	5	4
3	1	2
4	3	3
5	6	4
6	0	5
7	4	1

x	$g(x)$	$g'(x)$
0	5	5
1	7	1
2	4	4
3	2	6
4	6	10
5	3	4
6	1	2
7	0	1

(a) The function Q is defined by $Q(x) = x^2 g(x) - 5f(x)$. Which of the following is a stationary point of Q?

(A) 0 **(B)** $\boxed{1}$ **(C)** 2 **(D)** 3 **(E)** 4

(b) The function H is defined by $H(x) = f(x^2)$. Find $H'(2)$.

(A) -8 **(B)** 0 **(C)** 4 **(D)** 8 **(E)** $\boxed{12}$

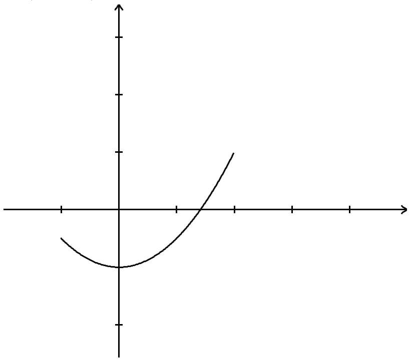
9c. The function G is defined by $G(x) = (f(x) + g(x))^2$. What is $G'(3)$?

(A) 12 **(B)** 16 **(C)** 24 **(D)** 32 **(E)** $\boxed{48}$

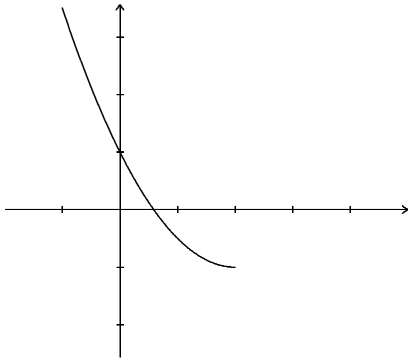
On all the following questions, **show your work.**

10. (20 points) On the coordinate axes provided sketch the graphs of four functions. In (a), sketch the graph of a function f that is increasing and concave upwards over the interval $[0, 2]$. In (b), sketch the graph of a function g that is decreasing and concave upwards over the interval $[0, 2]$. In (c), sketch the graph of a function h that is increasing on both $[0, 1]$ and on $(1, 2]$ and discontinuous at $x = 1$. In (d), sketch the graph of a continuous function k has a singular point at $x = 1$, and for which $k'(x) < 0$ for $0 < x < 1$ and $k'(x) > 0$ for $1 < x < 2$.

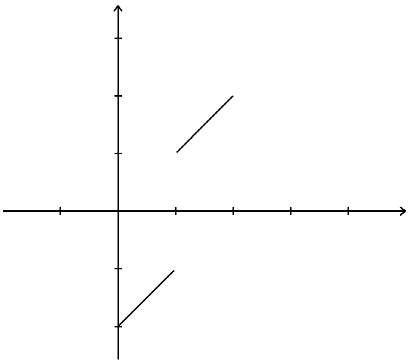
(a)



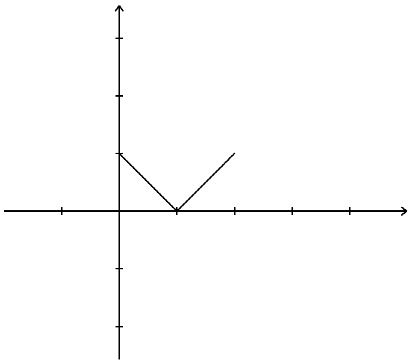
(b)



(c)



(d)



11. (20 points) Let $f(x) = x^3 - 6x^2 + 9x + 5$ on the interval $[-1, 5]$.

   (a) Find $f'(x)$ and $f''(x)$.

   Solution. $f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$.

   (b) Find all stationary and all singular points of f.

   Solution. Since f is a polynomial, it has no singular points. f has stationary points at all x for which $3x^2 - 12x + 9 = 0$. Factor this to get $3(x-3)(x-1) = 0$, so $x = 1$ and $x = 3$.

   (c) Find all values of x where f has relative extrema.

   Solution. f can have relative extrema only at stationary points, $x = 1$ and $x = 3$.

   (d) Use the second derivative test to identify each (c) as the location of a relative max or a relative min.

   Solution. Note that $f''(1) = 6 - 12 = -6 < 0$, so f has a relative max at $x = 1$. and $f''(3) = 18 - 12 = 6 > 0$, so f has a relative min at $x = 3$.

   (e) Find a point of inflection of f.

   Solution. $f''(x) = 0$ happens precisely when $6x - 12 = 0$, and f'' changes signs at $x = 2$, so the point $(2, f(2)) = (2, 7)$ is a point of inflection.

   (f) Find the absolute maximum and absolute minimum of f. You must show why all the other candidates were rejected to get full credit.

   Solution. Compare the values $f(-1) = -11$, $f(1) = -1$, $f(3) = 5$ and $f(5) = 25$ to see that f has its absolute maximum value of 25 at 5 and its absolute minimum value of -11 at -1.

12. (20 points) This problem was replaced on the test by the one above. Let $h(x) = \dfrac{(x-1)^2(2x+2)(x+4)}{(x-2)^2(x+3)(x-5)}$. Solve completely the inequality $h(x) > 0$.

    Solution. The 'critical' points of the function $h(x)$ are $x = 1, -1, -4, 2, -3$, and 5. Examine the sign of h at a test point in each of the intervals $(-\infty, -4)$, $(-4, -3)$, $(-3, -1)$, $(-1, 1)$, $(1, 2)$, $(2, 5)$ and $(5, \infty)$. You'll find that h is positive over $(-\infty, -4)$, $(-3, -1)$, and $(5, \infty)$.

13. (20 points) Optimal Charter Flight Fare. If exactly 300 people sign up for a charter flight, the agency charges \$250. However, if more than 300 sign up, the agency reduces the fare by \$0.50 for each additional person.

    (a) Let x denote the number of passengers beyond 300. Construct the revenue function $R(x)$.

    Solution. $R(x) = (300 + x)(250 - 0.5x)$.

    (b) Find all the stationary points of your revenue function.

    Solution. $R'(x) = 1(250 - 0.5x) - 0.5(300 + x)$. Therefore the stationary point the solution of the equation $1(250-0.5x)-0.5(300+x) = 100-x = 0$, namely, $x = 100$.

    (c) What number of passengers results in the maximum revenue?

    Solution. Since the second derivative, $R''(x) = -1$, the stationary point $x = 100$ give a relative maximum. The optimal number of passengers is therefore, $300 + 100 = 400$.

    (d) What is the maximum revenue?

    Solution. $R(100) = (300 + 100)(250 - 100/2) =$ \$80,000.