November 30, 2016 **Name** ______________________________

The problems count as marked. The total number of points available is 146. Throughout this test, **show your work.**

1. (10 points) How long does it take an investment of $\$P$ at an annual rate of 8% to triple in value if compounding

   (a) takes place quarterly? Round your answer to the nearest tenth of a year.

   Solution: We need to solve the equation $3P = P\left(1 + \frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$ for t. Using logs, we get $t = \frac{\ln 3}{4\ln 1.02} \approx 13.87 \approx 13.9$ years.

   (b) takes place continuously? Round your answer to the nearest tenth of a year. As usual, no work shown, no credit!

   Solution: We need to solve the equation $3000 = 1000e^{0.08t}$ for t. Using logs, we get $t = \frac{\ln 3}{0.08} \approx 13.73 \approx 13.7$ years.

2. (12 points) Let $f(x) = e^{x^3-27x}$

   (a) Find an equation for the line tangent to $f(x)$ at the point $(0, f(0))$.

   Solution: Since $f'(x) = (3x^2 - 27)e^{x^3-27x}$, $p'(0) = -27$ and the line through $(0, 1)$ with slope -27 is $y - 1 = -27x$.

   (b) Find an interval over which $f(x)$ is decreasing.

   Solution: The two critical points are $x = -3$ and $x = 3$, and $f'(x) < 0$ in the interval $[-3, 3]$, so p is decreasing on $[-3, 3]$.

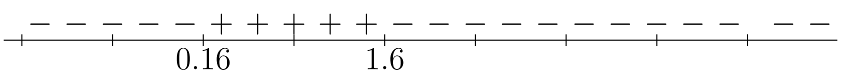
3. (20 points) Consider the function $f(x) = \ln(x^3 - x + 1)^{1/2}$.

   (a) Find $f'(x)$.
   Solution: $f'(x) = \frac{1}{2}\frac{3x^2-1}{x^3-x+1}$.

   (b) Find an equation for the line tangent to the graph of f at the point $(3, f(3))$.
   Solution: Since $f'(3) = \frac{1}{2}\frac{26}{25} = \frac{13}{25}$ and $f(3) = \frac{1}{2}\ln 25$, we have $y - \frac{1}{2}\ln 25 = 13(x-3)/25$.

   (c) Find $f''(x)$.
   Solution: $f''(x) = \frac{1}{2}\frac{6x(x^3-x+1)-(3x^2-1)^2}{(x^3-x+1)^2}$.

   (d) Find the sign chart for $f''(x)$.
   Solution: Using a graphing calculator, $f''(x) < 0$ on $(-\infty, 0.16)$ and on $(1.2, \infty)$ and positive on $(0.16, 1.2)$, as shown on the sign chart for f'':



   (e) Find the intervals over which f is concave upwards.
   Solution: From (c) it follows that f is concave upwards on $(0.16, 1.6)$.

4. (30 points) Let $H(x) = (x^2 - 9)^3(3x - 1)^2$.

   (a) Use the chain and product rules to find $H'(x)$.

   Solution:

$$\begin{aligned} H'(x) &= 3(x^2-9)^2 \cdot 2x(3x-1)^2 + 6(3x-1)\cdot(x^2-9)^3 \\ &= 6(x^2-9)^2(3x-1)[x(3x-1)+(x^2-9)] \\ &= 6(x-3)^2(x+3)^2(3x-1)[4x^2-x-9] \end{aligned}$$

   (b) Find the critical points of H.

   Solution: Thus, H has critical points $x = \pm 3, 1/3$ and $\frac{1+\sqrt{145}}{8}$. The last two are roughly $\alpha \approx -1.38$ and $\beta \approx 1.63$.

   (c) Build the sign chart for $H'(x)$

   Solution: $H'(x) \geq 0$ on $[\alpha, 1/3] \cup [\beta, \infty)$.

   (d) Classify the critical points of H as max, min, or imposters.

   Solution: Notice that $1/3$ is an maximum since $h'(x)$ is positive on the left side and negative on the right side. Then we have minimums at α and β imposters at both at -3 and 3.

   (e) Find the intervals over which H is increasing.

   Solution: From the sign chart for H', we see that H is increasing on $(\alpha, 1/3]$ and $(\beta], \infty)$.

5. (12 points) Polonium is a radioactive element which decays according to the law

$$Q(t) = Q_0 \cdot 2^{-t/140}.$$

where Q_0 is the initial amount and t is measured in days.

(a) If the amount of material left after 280 days is 20 mg, what was the initial amount?

Solution: Solve the equation $20 = Q_0 \cdot 2^{-2} = Q_0/4$ to get $Q_0 = 80$.

(b) What is half-life of Polonium?

Solution: Solve the equation $\frac{1}{2} = 2^{-t/140}$ to get $t = 140$ days.

(c) How much Polonium will be left after 400 days?

Solution: $Q(400) = 80 \cdot 2^{-400/140} = 80 \cdot 2^{-20/7} \approx 11.0$

6. (15 points) Living with HIV. On the basis of data from WHO, the number of people living with HIV worldwide from 1985 through 2006 is given by the logistic curve

$$N(t) = \frac{40}{1 + 19e^{-0.30t}}.$$

where $N(t)$ is measured in millions and t is measured in years, with $t = 0$ corresponding to the beginning of 1985.

(a) How many people were living with HIV at the beginning of 1985?

Solution: $N(0) = \frac{40}{1+19} = 2$ million.

(b) Assuming the trend continues, how many people living with HIV would we expect at the beginning of 2010?

Solution: $N(25) = \frac{40}{1+19e^{0.3(25)}} \approx 39.6$ million.

(c) At what rate is the disease growing at the beginning of the year 2000? Be sure to include your units in the answer.

Solution: $N'(t) = 40(-1)(1 + 19e^{-0.3t})^{-2}(19e^{-0.3t})(-0.3) = \frac{12 \cdot 19e^{-0.3t}}{(1+19e^{-0.3t})^2}$, so $N'(15) = \frac{12 \cdot 19e^{-4,5}}{(1+19e^{4.5})^2} \approx 2.1$ million per year.

7. (20 points) Demand for IPhones. Apple economists propose that the monthly demand for IPhone 7's is modeled by

$$D(t) = 20 - 15e^{-kt},$$

where t is measured in months and $D(t)$ is measured in millions of phones. Suppose that 6 million phones are sold during the third month of the year ($t = 3$).

(a) How many phones can Apple expect to sell during the fourth month of the year?

Solution: Solve $D(3) = 5$ to get $k = \frac{\ln(15)-\ln(14)}{3} \approx 0.023$. From this it follows that $D(4) \approx 6.32$ million.

(b) What is the demand after a year?

Solution: $D(12) = 20 - 15e^{-12k} \approx 8.6$ million.

(c) At what level of sales is demand expected to stabilize?

Solution: The line $y = 20$ is a horizontal asymptote.

(d) Explain why this model does not work for items like IPhone 7's.

Solution: Among lots of other reasons is the fact that Iphones are much more popular when they are released than they are later.

8. (15 points) Let $f(x) = \ln[(x^2+1)(x^3+x^2)(\frac{1}{x+2})]$.

   (a) Find the derivative of $f(x)$.

   Solution: First, rewrite f as $f(x) = \ln(x^2+1) + \ln((x^2)(x+1)) + \ln(x+2)^{-1} = \ln(x^2+1) + \ln(x^2) + \ln(x+1) + \ln(x+2)^{-1}$. At this point the differentiation is much easier: $f'(x) = \frac{2x}{x^2+1} + \frac{2}{x} + \frac{1}{x+1} - \frac{1}{x+2}$.

   (b) What is $f'(1)$?

   Solution: $f'(1) = 1 + 2 + 1/2 - 1/3 = 19/6$

   (c) Find an equation for the line tangent to f at the point $(1, f(1))$. Leave your answer in terms if the ln function.

   Solution: Since $f(1) = \ln(2 \cdot 2 \cdot 1/3) = \ln(4/3) = \ln(4) - \ln(3)$, we have $y - \ln(4/3) = 19(x-1)/6$ and the slope-intercept form is easy to get.

9. (20 points) A botanist conjectures that the height of a certain type of pine tree can be modeled by a learning curve. To test his conjecture, he plants a 2 foot tall tree. He knows that eventually the tree will grow to 40 feet tall, its maximum height. Suppose that after one year, the tree is 4 feet tall.

   (a) What does the model predict for the height of the tree after two years.

   Solution: We use the model $Q(t) = A - Be^{-kt}$ with the information that $Q(0) = 2$ and $\lim_{t\to\infty} Q(t) = 40$. So $A - B = 2$ and $A = 40$. Conclude that $B = 38$.

   (b) How many inches does the tree grow during the fourth year?

   Solution: Since the tree grows to 4 feet after one year, we have $Q(1) = 4 = 40 - 38e^{-k}$ which we solve for k to get $k = \ln(19) - \ln(18) \approx 0.05406$. So the number of inches grown during the fourth year is $Q(4) - Q(3) = 38(e^{-3k} - e^{-4k}) \approx 1.7$ feet, or 20.4 inches.

   (c) What is the instantaneous rate of growth at $t = 3.5$ years.

   Solution: To find the instantaneous rate of growth at $t = 3.5$ years, differentiate Q. $Q'(t) = 38ke^{-kt}$ and at $t = 3.5$ is 1.7003 feet or 20.40 inches.

   (d) Describe the connection between the two answers (b) and (c).

   Solution: These two quantities are quite close because $(Q(4)-Q(3)/(4-3)$ is a good estimate of $Q'(3.5)$. Look at the graph to see just how close these two are.