March 21, 2001 **Name** ________________________

The first 11 problems are true-false problems that count 3 points each. The rest are counted as marked. The total value of the test is 125.

True-false section. Circle the correct choice. You do not need to show your work on these problems.

1. True or false. If f and g are differentiable and a and b are constants, then $\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$.

   Solution: True. This is just the rule that talks about the derivative of the sum and of a constant times a function.

2. True or false. If $f'(x) > 0$ for each x in the interval $(-1, 1)$, then f is increasing on $(-1, 1)$.

   Solution: True.

3. True or false. If $f''(x) < 0$ on the interval (a, c) and $f''(x) > 0$ on the interval (c, b), then the point $(c, f(c))$ is a point of inflection of f.

   Solution: True.

4. True or false. If $f(a) < 0, f(b) > 0$, and $f'(x) > 0$ for each x in (a, b), then there is one and only one number c in (a, b) such that $f(c) = 0$.

   Solution: True. The Intermediate Value Theorem guarantees that there is at least one c in (a, b), and the condition $f'(x) > 0$ for each x in (a, b) guarantees that there can be no more than 1 such point.

5. True or false. The graph of a function cannot touch or intersect a horizontal asymptote to the graph of f.

   Solution: False. There is nothing in the definition of horizontal asymptote that implies this.

6. True or false. If $f'(c) = 0$, then f has a relative maximum or a relative minimum at $x = c$.

   Solution: False. The function can have neither a max nor a min at a stationary point. Look at $f(x) = x^3$ and 0.

7. True or false. If f has a relative maximum or a relative minimum at $x = c$, then $f'(c) = 0$.

   Solution: False. All we can tell is that c is a critical point. It might be a singular point.

8. True or false. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.

   Solution: True. This is just the second derivative test.

9. True or false. If f and g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.

   Solution: False. Look up the product rule.

10. True or false. If f and g are differentiable, then $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)}{g'(x)}$.

    Solution: False. Look up the quotient rule.

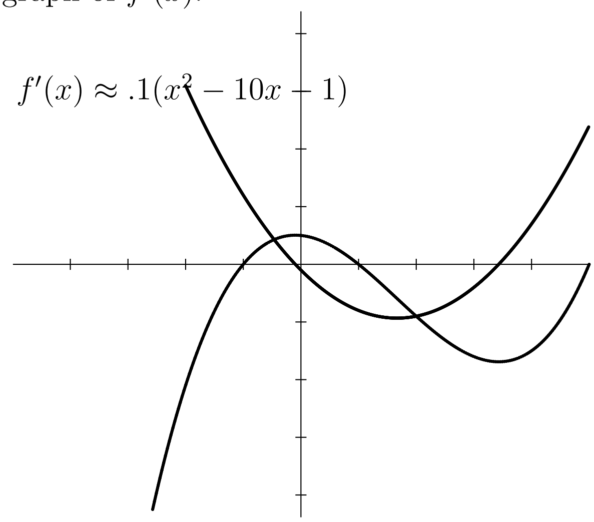
11. True or false. If f and g are differentiable and $h(x) = f \circ g$, then $h'(x) = f[g(x)]g'(x)$.

    Solution: False. Look up the chain rule.

12. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x) = x^3 - 4x^2 - x + 4$ on the interval $-2 \leq x \leq 6$.

    Solution: Note that $f'(x) = 3x^2 - 8x - 1$, so there are two critical points inside the interval $[-2, 6]$, lets call them $\alpha = \frac{8+\sqrt{76}}{6} \approx 2.786$ and $\beta = \frac{8-\sqrt{76}}{6} \approx -0.119$. We must compare the values $f(\alpha) \approx -8.208$, $f(\beta) \approx 4.0606$, $f(-2) = -18$, and $f(6) = 70$. Clearly, the largest and smallest values of f occur at the endpoints, 6 and -2 respectively.

13. (12 points) Let f be the function whose graph is shown below. On the same axes, plot the graph of $f'(x)$.



14. (12 points) Find the interval(s) where $f(x) = x^3 - 6x^2 - 4x + 8$ is increasing.

    Solution: Find $f'(x)$ and determine the critical points of f. $f'(x) = 3x^2 - 12x - 4$, and by the quadratic formula, the critical points are $\alpha = \frac{12+\sqrt{144+48}}{6} \approx 4.309$, and $\alpha = \frac{12+\sqrt{144+48}}{6} \approx -.309$. Because f is cubic with positive coef of x^3, it follows that f is increasing on $(-\infty, \beta]$ and on$[\alpha, \infty)$.

15. (12 points) Find the relative maxima and relative minima, if any, of $g(x) = x^2 + \frac{16}{x^2}$.

    Solution: First note that $g'(x) = 2x - 32x^{-3}$, which has value zero when $x^{-4} = 1/16$, ie, when $x = \pm 2$. Examining either the graph or the second derivative at these two points reveals that they are both locations of relative minimums, and that $g(-2) = g(2) = 8$.

16. (12 points) Let $f(x) = x^4 + 2x^3 - 12x^2 + 6x$.

    (a) Find the interval(s) where f is concave upward and the interval(s) where f is concave downward. Use the Test Interval technique to determine the places where f'' is positive and where it is negative.

    Solution: Find f' and f''. $f'(x) = 4x^3 + 6x^2 - 24x + 6$ and $f''(x) = 12x^2 + 12x - 24 = 12(x^2 + x - 2) = 12(x + 2)(x - 1)$, so there are two places where concavity COULD change. In fact the test interval technique applied to f'' shows that $f''(x) > 0$ on $(-\infty, -2)$ and on $(1, \infty)$. Thus f is concave up on these two intervals and down on $[-2, 1]$.

    (b) Find the inflection points of f, if there are any.

    Solution: There are two points of inflection, $(-2, f(-2) = (-2, -60)$ and $(1, -3)$.

17. (12 points) Consider the rational function

$$f(x) = \frac{(2x^2 - 3)(x - 2)}{(x^2 - 4)(x + 1)}.$$

    (a) Find the horizontal asymptotes.

    Solution: The coefficient of x^3 in the numerator is 2 while that in the denominator is 1, so $y = 2/1$ is the horizontal asymptote.

(b) Find the vertical asymptotes.

Solution: To find the vertical asymptotes, you must first reduce the fraction to lowest terms, which mean cancelling out the common factors, in this case, just the $x-2$'s. This results in a denominator that has value 0 only at $x = 1$ and $x = 2$, so these are the two vertical asymptotes.

(c) Compute $\lim_{x \to -\infty} f(x)$.

Solution: The limit in question is the same as the horizontal asymptote, 2.

On all the following questions, **show your work.**

18. (20 points) The quantity demanded per month, x of a certain brand of electric shavers is related to the price, p, per shaver by the equation $p = -0.1x + 10,000 \quad (0 < x < 20,000)$, where p is measured in dollars. The total monthly cost for manufacturing the shavers is given by $C(x) = 0.00002x^3 - 0.4x^2 + 10,000x + 20,000$. Construct the revenue function, $R(x)$. How is the profit related to revenue and cost? Find $P'(x)$, where $P(x)$ denotes the profit function. How many shavers should be produced per month in order to maximize the company's profit? What is the maximum profit?

    Solution: First, the revenue function is $R(x) = x \cdot p(x) = x(-0.1x + 10,000)$ and the profit function is given by $P(x) = R(x) - C(x)$. Thus $P(x) = x(-0.1x + 10,000) - (0.00002x^3 - 0.4x^2 + 10,000x + 20,000)$, and $P'(x) = -0.2x + 10,000 - 0.00006x^2 + 0.8x - 10,000$. Combining terms and simplifying yields $P'(x) = 0.6x - 0.00006x^2$, which leads to the critical point $x = 10,000$, and a maximum profit of $9,980,000$.