Math 1120, Section 6 **Calculus** **Test 3**

April 27, 2006 **Name** ______________________________

The total number of points available is 135. Throughout this test, **show your work.**

1. (8 points) How long does it take a 6% investment, compounded continuously, to triple in value?

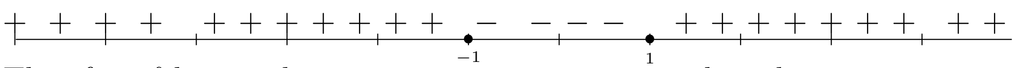
   Solution: Solve the equation $Pe^{rt} = 3P$ for t where P doesn't matter and $r = 0.06$. You get $e^{0.06t} = 3$ and finally $t = \ln(3)/0.06 \approx 18.31$ years.

2. (20 points) Consider the function $f(x) = e^{2x^3-6x}$.

   (a) Compute $f'(x)$.
   Solution: $f'(x) = (6x^2 - 6)e^{2x^3-6x}$.

   (b) Find the critical points of f.
   Solution: $(6x^2 - 6) = 6(x - 1)(x + 1) = 0$ at $x = 1$ and $x = -1$.

   (c) Find the intervals over which f is increasing.
   Solution: The sign chart for f' is



   Therefore f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

   (d) Compute $f''(x)$ and discuss the concavity of f. You will have to use your graphing calculator for this part of the problem.
   Solution: The second derivative of f is given by the product rule: $f''(x) = 12e^{2x^3-6x}(3x^4 - 6x^2 + x + 3)$. Use the trace feature on the graphing calculator to find that there are two zeros of f'', roughly $x = -1.28$ and $x = -0.73$. f'' is negative between these two values, and positive otherwise. Thus, f is concave down on $(-1.28, -0.73)$ and concave up otherwise.

3. (15 points) Certain radioactive material decays in such a way that the mass remaining after t years is given by the function

$$m(t) = 110e^{-0.02t}$$

where $m(t)$ is measured in grams.

(a) Find the mass at time $t = 0$.
Solution: $m(0) = 110e^{-0.02 \cdot 0} = 110$ grams.

(b) How much of the mass remains after 20 years?
Solution: $m(20) = 110e^{-0.02 \cdot 20} \approx 73.735$ grams.

(c) At what rate is the mass declining after 10 years?
Solution: $m'(t) = (-0.02)110e^{-0.02t}$, so $m'(10) = (-0.02)110e^{-0.02 \cdot 10} \approx -1.801$ grams per year.

4. (10 points) The population of the world in 1987 was 5 billion and the relative growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2007.

Solution: Note that $Q(t) = Q_0 e^{kt}$ and that $Q(0) = Q_0 = 5$ billion, so $Q(20) = 5e^{0.02 \cdot 20} = 5e^{0.4} \approx 7.459$ billion.

5. (16 points) A study finds that the average student taking advanced shorthand progresses according to the function

$$Q(t) = 120(1 - e^{-0.05t}) + 60, \qquad (0 \leq t \leq 20),$$

where $Q(t)$ measures the number of words (per minute) of dictation that the student can take in machine shorthand after t weeks in the twenty-week course. Sketch the graph of Q and answer the following questions:

(a) What is the beginning shorthand speed for the average student?
Solution: $Q(0) = 120(1 - e^{-0.05 \cdot 0}) + 60 = 60$ words per minute.

(b) What shorthand speed does the average student attain halfway through the course?
Solution: $Q(10) = 120(1 - e^{-0.05 \cdot 10}) + 60 = 107.2$ words per minute.

(c) How many words per minute can the average student take at the end of the course.
Solution: $Q(20) = 120(1 - e^{-0.05 \cdot 20}) + 60 = 135.8$ words per minute.

(d) What is the rate of change of the speed after exactly 5 weeks in the course?
Solution: $Q'(t) = 120(0 - e^{-0.05t} \cdot (-0.05))$, so $Q'(5) = 6(e^{-0.05 \cdot 5}) = 6e^{-.25} \approx 4.67$ words per minute per week.

6. (12 points) Let $f(x) = 3\ln(x^2 - 3)$. Notice that the domain of f includes the point $x = 2$, since $2^2 - 3 = 1 > 0$.

(a) Find $f'(x)$.
Solution: $f'(x) = 3\left(\frac{2x}{x^2-3}\right) = \frac{6x}{x^2-3}$.

(b) Find $f'(2)$.
Solution: $f'(2) == \frac{12}{4-3} = 12$.

(c) Find an equation for the line tangent to the graph of f at the point $(2, f(2))$.
Solution: $y - 0 = 12(x - 2)$ so $y = 12x - 24$.

7. (16 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the difference between the object's temperature and that of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that
$$F(t) = T + Ae^{-kt},$$
where T is the air temperature, $70°F$, A and k are constants, and t is expressed in minutes.

   (a) What is the value of A?

   Solution: Note that $F(0) = 70 + A \cdot 1 = 212$ so $A = 142$.

   (b) Suppose that after exactly 20 minutes, the temperature of the coffee is $186.6°F$. What is the value of k?

   Solution: Solve $F(t) = 186.6 = 70+142e^{-k(20)}$ for k to get $k \approx 0.009853$.

   (c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

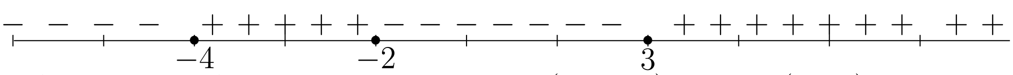
   Solution: Solve the equation $80 = 70 + 142e^{-0.009853t}$ for t to get first $e^{-0.009853t} = 10/142 \approx 0.0704$, and taking logs of both sides yields $t = 269.28$ minutes.

   (d) Find the rate at which the object is cooling after $t = 20$ minutes.

   Solution: To find $F'(t)$ recall the way we differentiate exponential functions. $F'(t) = 142(-k)e^{-kt}$, so $F'(20) = 140(-k)e^{-20k} \approx -1.1488$ degrees per minute.

8. (10 points) Suppose the function $f(x)$ has been differentiated twice to get $f''(x) = (x - 3)(x + 2)(x + 4)$. Find the intervals over which $f(x)$ is concave upward.

   Solution: The branch points are $x = 3, -2$, and $x = -4$. The sign chart for f'' is given below:



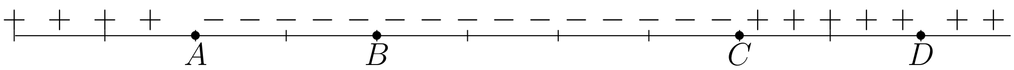
It follows that f is concave upwards on $(-4, -2)$ and on $(3, \infty)$.

9. (12 points) Find the critical points of each function.

   (a) $f(x) = (x^2 - 4)^2(2x - 3)^2$
   Solution: Use the product rule to get $f'(x) = 2(x^2-4)\cdot 2x \cdot (2x-3)^2 + 2(2x-3)^2 \cdot 2(x^2-4)^2 = 4(x^2-4)(2x-3)[x(2x-3)+(x^2-4)] = 4(x^2-4)(2x-3)[3x^2-3x-4]$, so the critical points are $x = \pm 2$, $x = 3/2$, and the two obtained from the quadratic formula $x = \frac{3\pm\sqrt{9+4\cdot3\cdot4}}{6} \approx 1.758, -0.758$.

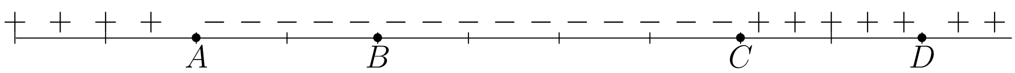
   (b) $g(x) = (x^2 - 16)^{2/3}$
   Solution: $f'(x) = 2[(x^2-16)^{-1/3} \div 3]\cdot 2x$ so there are critical points of both types, singular and stationary. The stationary critical point is $x = 0$ because it makes the numerator zero and the two singular points (which make the denominator zero) are the x values that make f' undefined, namely $x = 4$ and $x = -4$.

10. (16 points) Given below is a sign chart for the derivative $f'(x)$ of a function.

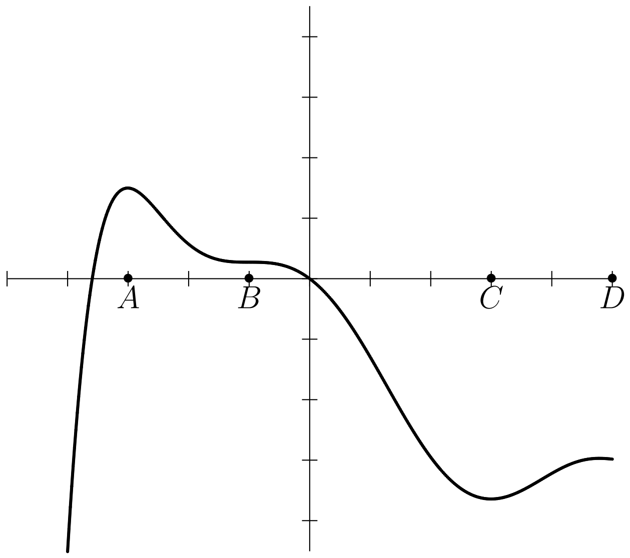


   (a) For each of the stationary points A, B, C and D tell whether $f(x)$ has a relative maximum, relative minimum, or neither at the point.
   Solution: Since f is increasing to the left of A and decreasing just to the right of A, it must have a local max at A. Since f' has the same sign on both sides of B and D, it has neither a max nor a min at these points. It has a relative minimum at C.

   (b) Suppose $f(x)$ is a polynomial function with critical points A, B, C and D. Sketch a function on the coordinate system below that could have a derivative whose sigh chart is the one given.



   Solution: One such graph is given below.


A
B
C
D