**October 24, 2013** **Name** ______________________

The problems count as marked. The total number of points available is 171. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (36 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. Find the critical points for each function and the intervals over which the function is increasing. You must show your work.

   (a) Let $F(x) = (2x + 8)(4x - 6)$

   **Solution:** Note that $F'(x) = 16x + 20$, so the only critical point is $x = -20/16$. Since $f'(x) > 0$ on $(-20/16, \infty)$, we conclude that $F$ is increasing on that interval.

   (b) $G(x) = \dfrac{x^2 - 3x + 15/2}{2x - 1}$

   **Solution:** By the quotient rule, $G'(x) = \frac{(2x-3)(2x-1)-2(x^2-3x+15/2)}{(2x-1)^2} = \frac{2x^2-2x-12)}{(2x-1)^2}$. So the critical points are $x = -2$ and $x = 3$. We can see that $G' > 0$ on both $(-\infty, -2)$ and on $(3, \infty)$. So $G$ is increasing on these two intervals.

   (c) $K(x) = (x^2 - 4)^{18}$

   **Solution:** By the chain rule, $K'(x) = 18(x^2 - 4)^{17} \cdot 2x$, so the critical points are $x = -2, 0, 2$. Build the sign chart for $K'$ to see that $K'$ is positive on both $(-2, 0)$ and $(2, \infty)$. Therefore $K$ is increasing on those two intervals.

2. (10 points) The line tangent to the graph of $g(x)$ at the point $(4, 6)$ has a $y$-intercept of 9. What is $g'(4)$?

   **Solution:** The line has slope $(9 - 6)/(0 - 4) = -3/4$.

3. (10 points) Find all the points $(x, y)$ on the graph of $h(x) = 2x^2 - 4x$ where the tangent line has a slope equal to 5.

   **Solution:** Since $h'(x) = 4x - 4$, we can solve $4x - 4 = 5$ for $x$, which yields $x = 9/4$ and $y = 2 \cdot 81/16 - 36/4 = 18/16 = 9/8$.

4. (15 points) Show that the function $f(x) = 7\sqrt{x} - x^3$ has a critical point in the interval $[1, 4]$. Note that both $f$ and $f'$ are continuous on the interval $[1, 4]$.

   **Solution:** Compute $f'$: $f'(x) = 7 \cdot \frac{1}{2}x^{-1/2} - 3x^2$. Next, $f'(1) = 7/2 - 3 > 0$ while $f'(4) = 7/4 - 48 < 0$. Applying the Intermediate Value Theorem to the continuous function $f'(x)$, we conclude that $f'$ has a zero in the interval $[1, 4]$.

5. (10 points) The line tangent to the graph of a function $f$ at the point $(2, 9)$ on the graph also goes through the point $(0, 7)$. What is $f'(2)$?

   **Solution:** The slope of the line through $(2, 9)$ and $(0, 7)$ is 1, so $f'(2) = 1$.

6. (10 points) Find an equation for the line tangent to the graph of $f(x) = x^2 - 3x$ at the point (2,-2)?

   **Solution:** The derivative is $f'(x) = 2x - 3$ whose value of at $x = 2$ is $f'(2) = 4 - 4 = 1$. Thus the line is $y - (-2) = 1(x - 2)$, which, in slope intercept form is $y = x - 4$.

7. (15 points) Discuss the concavity of the function $r(x) = \frac{2x-3}{x+1}$.

   **Solution:** Note that $r'(x) = \frac{2(x+1)-1(2x-3)}{(x+1)^2}$ and $r''(x) = \frac{d}{dx}5(x + 1)^{-2} = -10(x+1)^{-3}$, which changes sign at $x = -1$ from negative to positive. Therefore, $r$ is concave upwards on the interval $(-1, \infty)$.

8. (35 points) Consider the table of values given for the functions $f, f', g$, and $g'$:

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 2 | 1 | 3 | 2 |
| 1 | 4 | 6 | 2 | 5 |
| 2 | 6 | 4 | 3 | 4 |
| 3 | 1 | 2 | 5 | 3 |
| 4 | 3 | 5 | 2 | 6 |
| 5 | 5 | 3 | 4 | 1 |
| 6 | 0 | 3 | 1 | 4 |

(a) $Q(x) = f(x)/g(x)$. Find $Q(5)$ and $Q'(5)$.

**Solution:** First, $Q(5) = f(5)/g(5) = 5/4$. $Q'(x) = (f'(x)g(x) - g'(x)f(x))/(g(x))^2$. Therefore, $Q'(5) = (f'(5)g(5) - g'(5)f(5))/(g(5))^2 = \frac{3\cdot 4 - 1\cdot 5}{4^2} = \frac{7}{16}$.

(b) Let $H(x) = f(x) \cdot g(x+1)$. Compute $H(4)$ and $H'(4)$.

**Solution:** $H(4) = f(4) \cdot g(5) = 3 \cdot 4 = 12$. By the product and chain rules, $H'(x) = f'(x) \cdot g(x+1) + g'(x+1) \cdot f(x)$. Therefore, $H'(4) = f'(4) \cdot g(4+1) + g'(5) \cdot f(4) = 5 \cdot 4 + 1 \cdot 3 = 23$.

(c) Let $W(x) = f(g(x) + 1)$. Compute $W(5)$ and $W'(5)$.

**Solution:** Again by the chain rule, $W'(x) = f'(g(x)+1) \cdot g'(x)$, so $W'(5) = f'(g(5)+1) \cdot (g'(5)) = 3 \cdot 1 = 3$.

(d) Let $L(x) = g(\frac{1}{x} + 1)$. Compute $L(1)$ and $L'(1)$.

**Solution:** By the chain rule, $L'(x) = g'(\frac{1}{x} + 1) \cdot (-x^{-2})$, so $L'(2) = g'(\frac{1}{1} + 1) \cdot (-1^{-2}) = g'(2) \cdot (-1) = -4$.

(e) Let $U(x) = g(x^2 - 2) + f(x)$. Compute $U(2)$ and $U'(2)$.

**Solution:** First, $U(2) = g(2^2 - 2) + f(2) = 3 + 6 = 9$ By the chain rule, $U'(x) = g'(x^2-2) \cdot 2x + f'(x)$, so $U'(2) = g'(2) \cdot 2 \cdot 2 + f'(2) = 4 \cdot 2 \cdot 2 + 4 = 20$.

(f) Let $Z(x) = g(2x - f(x))$. Compute $Z(4)$ and $Z'(4)$.

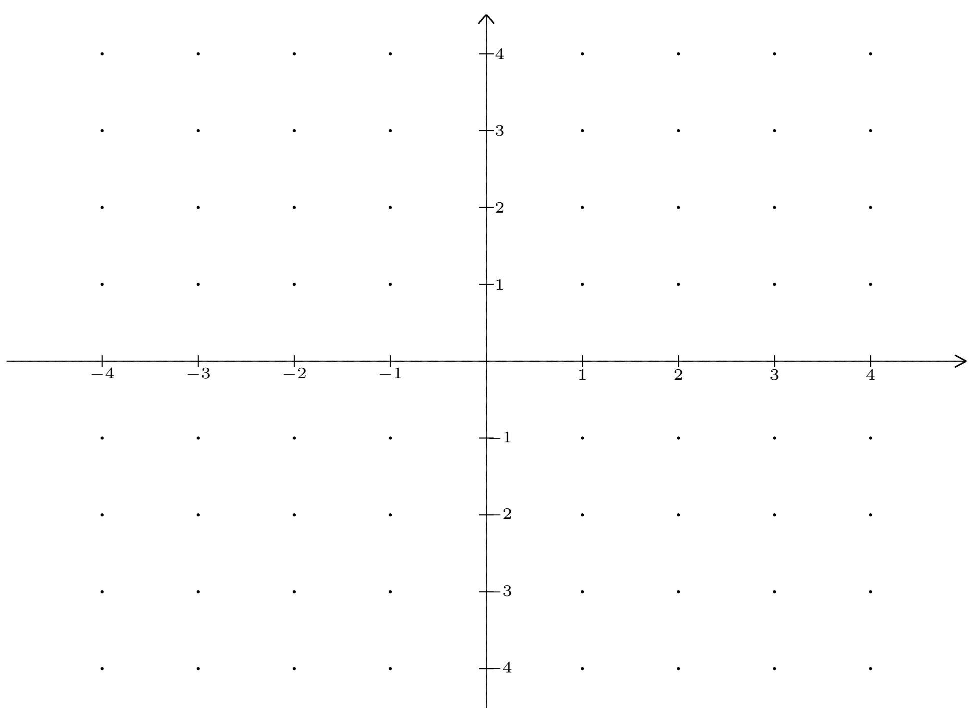
**Solution:** Again by the chain rule and the product rule, $Z'(x) = g'(2x - f(x)) \cdot (2 - f'(x))$ so $Z'(4) = g'(8 - f(4)) \cdot (2 - f'(4)) = g'(5)(2-5) = 1(-3) = -3$.

9. (30 points) Consider the function

$$r(x) = \frac{(x^2 - 4)(6x)}{(3x - 6)(x + 1)(x - 3)}.$$

Use the Test Interval Technique to find the sign chart of $r(x)$. Find the zeros and the horizontal and vertical asymptotes, and sketch the graph of $r$. Your graph must be consistent with the information you find in the sign chart.

**Solution:**




**Solution:** Notice first that $r$ is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x - 2)(x + 2)(6x)}{3(x - 2)(x - 3)(x + 1)}.$$

We can remove the common factor $x - 2$ with the understanding that we are (very slightly) enlarging the domain of $r$: $r(x) = \frac{(x+2)(6x)}{3(x-3)(x+1)}$. Next find the branch points. These are the points at which $r$ can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $0, -1, -2, 3$. The horizontal asymptote is $y = 6/3 = 2$, the vertical asymptotes are $x = 3$ and $x = -1$ and the zeros of $r$ are $x = 0$ and $x = -2$. Again we select test points and find the sign of $f$ at of these points to get the sign chart.