**February 27, 2013** **Name** ______________________________

The total number of points available is 151. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (12 points) Let $H(x) = \sqrt{x^2 - 2x + 4}$.

   (a) Find two functions, $f$ and $g$ whose composition $f \circ g$ is $H$, and use the chain rule to find $H'(x)$

   **Solution:** First, let $f(x) = x^{1/2}$ and let $g(x) = x^2 - 2x + 4$. Then, by the chain rule, $H'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2}(x^2 - 2x + 4)^{-1/2} \cdot (2x - 2) = \frac{2x-2}{2\sqrt{x^2-2x+4}}$.

   (b) What is $H'(2)$?

   **Solution:** $H'(2) = \frac{4-2}{2 \cdot 4^{1/2}} = 1/2$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of $H$ at the point $(2, H(2))$.

   **Solution:** Since $H(2) = \sqrt{4} = 2$, using the point-slope form leads to $y - 2 = H'(2)(x - 2) = (x - 2)/2$, so $y = x/2 + 1$.

2. (10 points) Solve the inequality $x^2 - 13x + 14 \leq 2$. Write your answer in interval notation.

   **Solution:** Rewrite the inequality as $x^2 - 13x + 12 \leq 0$, so $(x - 12)(x - 1) \leq 0$. Now the sign chart for $(x - 12)(x - 1)$ shows that the function is at most zero on $[1, 12]$.

3. (12 points) Consider the function $f$ defined by:

$$f(x) = \begin{cases} \sqrt{x+8} & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3(x-2)^2 & \text{if } x > 1 \end{cases}$$

   (a) Is $f$ continuous at $x = 1$? Your answer must make clear that you know and understand the definition of continuity. A yes/no correct answer is worth 1 point.

   **Solution:** No, the limits from the left and right are both 3, but the value of $f$ at 1 is 2, so $\lim_{x\to 1} f(x) \neq f(1)$.

   (b) What is the slope of the line tangent to the graph of $f$ at the point $(8, 108)$?

   **Solution:** To find $f'(8)$ first note that when $x$ is near 8, $f(x) = 3(x-2)^2$ so $f'(x) = 6(x-2)$. Thus, $f'(8) = 6(8-2) = 36$.

   (c) Find $f'(-2)$

   **Solution:** To find $f'(-2)$, we must differentiate the part of $f$ defined for $x < 1$. In this area, $f'(x) = (x+8)^{-1/2}/2$, so $f'(-2) = \frac{1}{2\sqrt{6}}$.

4. (12 points) Compute each of the following derivatives.

   (a) Let $f(x) = (2x+1)^2(x^2+x-1)$. Find $f'(x)$.

   **Solution:** Use the product rule to get $f'(x) = 2(2x+1)\cdot 2(x^2+x-1) + (2x+1)\cdot(2x+1)^2$.

   (b) Let $g(x) = \frac{x^2+x-1}{x^2+x+1}$. Find $g'(x)$.

   **Solution:** By the quotient rule, $g'(x) = \frac{(2x+1)(x^2+x+1)-(2x+1)(x^2+x-1)}{(x^2+x+1)^2} = \frac{4x+2}{(x^2+x+1)^2}$.

5. (18 points) If a ball is thrown vertically upward from the roof of 128 foot building with a velocity of 64 ft/sec, its height after $t$ seconds is $s(t) = 128 + 64t - 16t^2$.

   (a) What is the height the ball at time $t = 1$?

   **Solution:** $s(1) = 176$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   **Solution:** $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) What is the maximum height the ball reaches?

   **Solution:** Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 128 + 64(2) - 16(2)^2 = 192$.

   (d) After how many seconds is the ball exactly 160 feet above the ground?

   **Solution:** Use the quadratic formula to solve $128 + 64t - 16t^2 = 160$. You get $t = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$.

   (e) How fast is the ball going the first time it reaches the height 160?

   **Solution:** Evaluate $s(t)$ when $t = 2 - \sqrt{2}$ to get $32\sqrt{2}$.

   (f) How fast is the ball going the second time it reaches the height 160?

   **Solution:** Evaluate $s(t)$ when $t = 2 + \sqrt{2}$ to get $-32\sqrt{2}$. In other words the ball is going downward at the same rate it was moving upwards when first went through 160 feet.

6. (12 points) The cost of producing $x$ units of stuffed alligator toys is $C(x) = 0.004x^2 + 4x + 6000$.

   (a) Find the marginal cost at the production level of 1000 units.

   **Solution:** $C'(x) = \frac{d}{dx}(0.004x^2+4x+6000) = 0.008x+4$ so $C'(1000) = 12$.

   (b) What is the marginal average cost function?

   **Solution:** $\overline{C}(x) = 0.004x + 4 + 6000x^{-1}$, so $\overline{C}'(x) = 0.004 - 6000x^{-2}$.

   (c) What is $\overline{C}'(500)$? Interpret your answer. In particular, what does the sign of $\overline{C}'$ at $x = 500$ tell you?

   **Solution:** $\overline{C}'(500) = -0.02$, which means the average cost is decreasing when the production level is 500.

7. (35 points) Consider the table of values given for the functions $f, f', g$, and $g'$:

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 2 | 1 | 3 | 2 |
| 1 | 4 | 6 | 2 | 5 |
| 2 | 6 | 4 | 3 | 4 |
| 3 | 1 | 2 | 5 | 3 |
| 4 | 3 | 5 | 2 | 6 |
| 5 | 5 | 3 | 4 | 1 |
| 6 | 0 | 3 | 2 | 4 |

(a) Let $L(x) = (f(x) + g(x))^2$. Compute $L(2)$ and $L'(2)$.

**Solution:** $L(2) = 81$ and $L'(x) = 2(f(x) + g(x))(f'(x) + g'(x))$, so $L'(2) = 2(f(2) + g(2))(f'(2) + g'(2)) = 2(6 + 3)(4 + 4) = 144$.

(b) Let $U(x) = f \circ f \circ f(x)$. Compute $U(1)$ and $U'(1)$.

**Solution:** First, $U(1) = f(f(f(1))) = f(3) = 1$. By the chain rule, $U'(x) = f'(f \circ f(x)) \cdot f'((f(x)) \cdot f'(x)$, so $U'(1) = f'(f \circ f(1)) \cdot f'((f(1)) \cdot f'(1) = f'(f((f(1)) \cdot f'(f(1)) \cdot f'(1) = f'(3) \cdot f'(4) \cdot f'(1) = 2 \cdot 5 \cdot 6 = 60$.

(c) Let $K(x) = g(x) + f(x^2)$. Compute $K(2)$ and $K'(2)$

**Solution:** $K(2) = g(2) + f(4) = 3 + 3 = 6$ and $K'(x) = g'(x) + f'(x^2)2x$, so $K'(2) = g'(2) + f'(2^2)2 \cdot 2 = 4 + 5 \cdot 4 = 24$.

(d) Let $Z(x) = 1/g(2x)$. Compute $Z(3)$ and $Z'(3)$.

**Solution:** $Z(3) = 1/g(6) = 1/2$. Rewriting $Z$ as $Z(x) = g(2x)^{-1}$, by the chain rule, we have $Z'(x) = -1g(2x)^{-2} \cdot g'(2x) \cdot 2$ so $Z'(3) = -1 \cdot (1/2)^{-2}(g(6))^2 \cdot 2 = -2$.

(e) Let $Q(x) = g(3x) + f(2x)$. Compute $Q(2)$ and $Q'(2)$.

**Solution:** First, $Q(2) = g(6) + f(4) = 2 + 3 = 5$. By the sum rule and chain rule, $Q'(x) = 3g'(3x) + 2f'(2x)$ so $Q'(2) = 3g'(6) + 2f'(4) = 3 \cdot 4 + 2 \cdot 5 = 22$.

8. (20 points) Find all critical points of $H(x) = (x+2)^3 \cdot (x^2-1)^2$. Then identify each critical point as the location of a local maximum, local minimum, or neither.

   **Solution:** By the product rule, $H'(x) = 3(x+2)^2 \cdot (x^2-1)^2 + 2(x+2)^3(x^2-1)(2x) = 2(x+2)^2(x^2-1)[3(x^2-1) + 2x \cdot 2(x+2)] = (x+2)^2(x^2-1)[3x^2 - 3 + 4x^2 + 8x]$, so the stationary points are $x = -2$ and $x = \pm 1$ and the zeros of $7x^2 + 8x - 3$. Using the quadratic formula, we find two roots, $\alpha = \frac{-4-\sqrt{37}}{7} \cong -1.44$ and $\beta = \frac{-4+\sqrt{37}}{7} \cong 0.29$ Since $H'(x)$ is negative to the left of $-2$, between $\alpha$ and $-1$ and from $\beta$ to 1, it follows that $H$ has a local max at both $\alpha$ and $\beta$ and relative minima at the critical points $-2, -1$, and 1.

9. (20 points) The purpose of this problem is to show how we can prove the power rule when the exponent is not a positive integer. In class we showed that when $n$ is a positive integer,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

But our proof does not work for fractional exponents. Let $g(x) = x^{1/4}$. We want to prove that $g'(x) = \frac{1}{4}x^{-3/4}$. To accomplish this, construct a function $f(x)$ so that the composition $f \circ g(x)$ can be differentiated using just the power rule with positive integer exponents. Several choices will work here. This part of the problem is worth 6 points. Let $h(x) = f \circ g(x)$. Then use the chain rule to write $h'(x) = f'(g(x)) \cdot g'(x)$. You can find both $h'(x)$ and $f'(x)$ easily, so you can solve for $g'(x)$. Do this to get the desired conclusion

$$g'(x) = \frac{1}{4}x^{-3/4}.$$

**Solution:** Let $f(x) = x^4$. Then $h(x) = (x^{1/4})^4 = x$, so $h'(x) = 1$. Note that $f'(x) = 4x^3$. Thus,

$$1 = h'(x) = f'(g(x)) \cdot g'(x) = 4(x^{1/4})^3 \cdot g'(x),$$

so $g'(x) = 1/(4x^{3/4}) = \frac{1}{4}x^{-3/4}$.

If 1600 square centimeters of material is available to make a box with a square base and an open top, ?nd the largest possible volume of the box. Volume = cubic centimeters.