March 5, 2009 **Name** ______________________________

The problems count as marked. The total number of points available is 131. Throughout this test, **show your work.**

1. (12 points) Consider the cubic curve $f(x) = 2x^3 + 3x + 2$.

   (a) What is the slope of the line tangent to the graph of f at the point $(0, 2)$?
   Solution: $f'(x) = 6x^2 + 3$ so $f'(0) = 3$. The slope of the tangent line is 3.

   (b) Write an equation of this tangent line in the form $y = mx + b$.
   Solution: Since the slope is 2, we can use the point slope form to get $y - 2 = 3(x - 0) = 3x$, so $y = 3x + 2$.

2. (10 points) Suppose f is a function satisfying $f(2) = 3$ and $f'(2) = -1$. What is the y-intercept of the line tangent to the graph of f at the point $(2, 3)$?

   Solution: The tangent line is given by $y - 3 = -1(x - 2)$, which is the same line as $y = -x + 5$, so the y-intercept is 5.

3. (15 points) For what values of x is the line tangent to the graph of

$$f(x) = (2x + 1)^2(3x - 4)^2$$

   parallel to the line $y = 7$?

   Solution: Since $f'(x) = 2(2x + 1) \cdot 2 \cdot (3x - 4)^2 + 2(3x - 4) \cdot 3 \cdot (2x + 1)^2$, we seek those values of x such that $2(2x + 1)(3x - 4)[2(3x - 4) + 3(2x + 1)] = 0$. Factoring and simplifying yields $x = -1/2, x = 4/3$ and $6x - 8 + 6x + 3 = 0$, so the three values of are $x = -1/2, x = 4/3$ and $x = 5/12$.

4. (20 points) If a ball is thrown vertically upward from the roof of 256 foot building with a velocity of 64 ft/sec, its height after t seconds is $s(t) = 256 + 64t - 16t^2$.

   (a) What is the height the ball at time $t = 1$?

   Solution: $s(1) = 256 + 64 - 16 = 304$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) At what time t does the ball reach its maximum height?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith.

   (d) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 256 + 64(2) - 16(2)^2 = 320$.

   (e) After how many seconds is the ball exactly 176 feet above the ground?

   Solution: Use the quadratic formula to solve $256+64t-16t^2 = 176$. You get $-16t^2 + 64t + 80 = 0$. The left side can be factored to get solutions $x = -1$(nonsense) and $x = 5$(yes!).

   (f) The second derivative $s''(t)$ of the position function, also called the *acceleration* function, is denoted $a(t)$. Compute $a(t)$. Explain why this function is negative for all values of t.

   Solution: $a(t) = s''(t) = \frac{d}{dt}v(t) = \frac{d}{dt}64-32t = -32$. Its negative because the force of gravity pulls downward.

   (g) How fast is the ball going the first time it reaches the height 176? Write the answer with the correct units.

   Solution: Using the solution $t = 5$ above, we find that the velocity at $t = 5$ is $v(5) = 64 - 32 \cdot 5 = -96$ feet per second.

   (h) How fast is the ball going when it hits the ground?

   Solution: The ball hits the ground when $s(t) = 0$ which, by the quadratic formula is $t = 2 \pm 2\sqrt{5}$. Of course, only the positive on of these is relevant. Now $v(2 + 2\sqrt{5}) = 64 - 32(2 + 2\sqrt{5}) = -64\sqrt{5} \approx -143$ feet per second. Its negative because the ball is moving downward.

5. (20 points) Let

$$g(x) = (2x^2 - 1)^2(6x).$$

   (a) Compute $g'(x)$.

   Solution: By the product rule, $g'(x) = 2(2x^2-1)(4x)\cdot 6x + 6\cdot(2x^2-1)^2 = (2x^2-1)[48x^2 + 6(2x^2-1)]$.

   (b) Find the critical points of $g(x)$.

   Solution: Solve $g'(x) = 0$ for the four values $x = \pm\sqrt{1/2} \approx \pm 0.707$ and $x = \pm\sqrt{1/10} \approx \pm 0.316$.

   (c) Build the sign chart for $g'(x)$.

   Solution: g' changes signs at each critical point.

   (d) Use the sign chart for $g'(x)$ to discuss the nature of each critical point. In other words tell whether each critical point is the location of a local maximum, a local minimum, or neither.

   Solution: Since g' is positive on $(-\infty, -0.707) \cup (-0.316, 0.316) \cup (0.707, \infty)$, it follows that g has maximas at -0.707 and 0.316 and minimas at the other two critical points.

6. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $V(x) = f(x) \cdot g(x))$. Compute $V'(5)$.
Solution: By the product rule, $V'(x) = f'(x)g(x)+g'(x)f(x)$, so $V'(5) = f'(5)g(5) + g'(5)f(5) = 3 \cdot 4 + 1 \cdot 5 = 17$.

(b) Let $W(x) = \frac{g(x)}{f(x)}$. Compute $W'(3)$.
Solution: By the quotient rule, $W'(x) = [g'(x)f(x) - f'(x)g(x)] \div f(x)^2$, so $W'(3) = [3 \cdot 1 - 2 \cdot 5] \div 1 = -7$.

(c) Let $L(x) = f(x+1) - g(x)^2$. Compute $L'(2)$.
Solution: By the chain rule, $L'(x) = (f'(x+1) - 2g(x)g'(x)$, so $L'(2) = (f'(2+1) - 2g(2)g'(2) = 2 - 2 \cdot 3 \cdot 4 = -22$.

(d) Let $U(x) = (f \circ g)(2x+1)$. Notice that this is a composition of three functions. Compute $U'(1)$.
Solution: By the chain rule, $U'(x) = f'(g(2x+1)) \cdot 2$, so $U'(1) = f'(g(3)) \cdot g'(3) \cdot 2 = 3 \cdot 3 \cdot 2 = 18$.

(e) Let $K(x) = g(x^3 + 2)$. Compute $K'(1)$
Solution: $K'(x) = g'(x^3+2) \cdot 3x^2$, so $K'(1) = g'(3) \cdot 3 = 3 \cdot 3 = 9$.

7. (24 points) Compute the following derivatives.

   (a) Let $f(x) = x^2 + x^{-\frac{2}{3}}$. Find $\frac{d}{dx}f(x)$.
   Solution: $\frac{d}{dx}f(x) = 2x - 2x^{-\frac{5}{3}}/3 = 2x - \frac{2}{3x^{\frac{5}{3}}}$.

   (b) Let $g(x) = \sqrt{x^2 + x + 4}$. What is $g'(x)$?
   Solution: $g'(x) = 1/2(x^3 + x + 4)^{-1/2} \cdot (3x^2 + 1) = \frac{3x^2+1}{2\sqrt{x^3+x+4}}$.

   (c) Find $\frac{d}{dx}((3x + 1)^2 \cdot (4x^2 - 1))$
   Solution: Let $f(x) = ((3x + 1)^2 \cdot (4x^2 - 1))$. Then, by the product and chain rules, $f'(x) = 2(3x+1)\cdot 3(4x^2-1)+8x(3x+1)^2 = (3x+1)(48x^2 + 8x - 6)$.

   (d) Find $\frac{d}{dt}\frac{2t^2-3t}{t^2-1}$
   Solution: Use the quotient rule to get $\frac{d}{dt}\frac{2t^2-3t}{t^2-t} = \frac{(4t-3)(t^2-t)-(2t-1)(2t^2-3t)}{(t^2-t)^2}$. Expanding and collecting like terms gives $\frac{t^2}{t^2(t-1)^2} = \frac{1}{(t-1)^2}$. The simple nature of the answer makes us have a look at the original function. Notice that t is a factor of both numerator and denominator. So why not just eliminate the t's in the function and find the derivative of $\frac{2t-3}{t-1}$. Indeed, differentiating this function is much easier.